**US Army Corps
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Waterways Experiment
Station

Working Forum on Manifold Method of Material Analysis, Volume 2

The Numerical Manifold Method and Simplex Integration

by   *Gen-hua Shi*

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The Numerical Manifold Method and Simplex Integration

by Gen-hua Shi

U.S. Army Corps of Engineers
Waterways Experiment Station
3909 Halls Ferry Road
Vicksburg, MS 39180-6199

Volume 2 of 2

**US Army Corps
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500   0   500 m

AREA OF RESERVATION : 2.7 sq km

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PREFACE
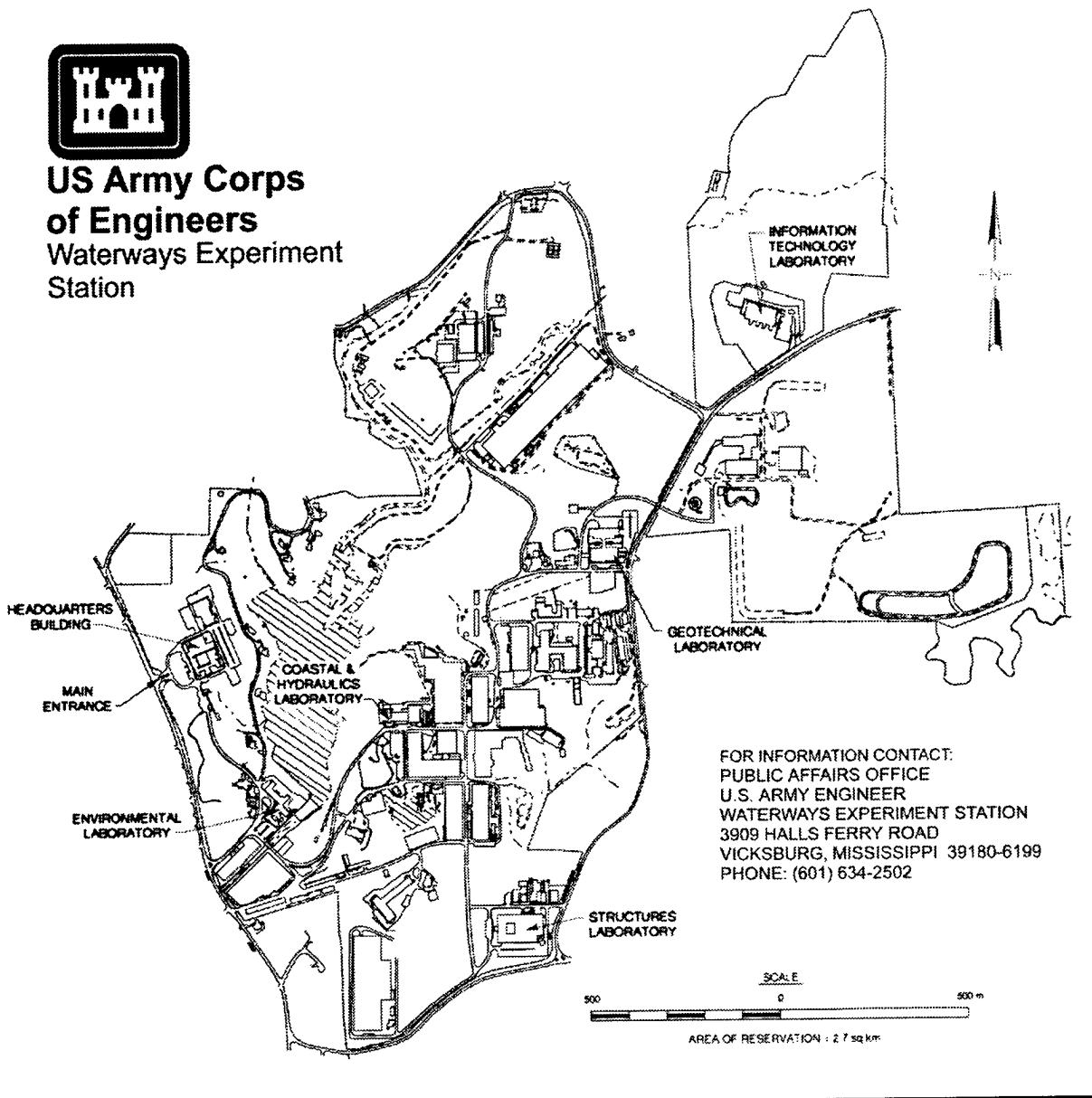
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The forum was conducted under the general supervision of Dr. W. F. Marcuson III, Director, GL. Dr. Don C. Banks, Chief, SRMD, provided direct supervision.

At the time of publication of this report, Dr. Robert W. Whalin was Director of WES.

Numerical Manifold Method

Gen-hua Shi

Geotechnical Lab, US Army Engineer Waterways Experiment Station
Vicksburg, MS 39180-6199

Abstract

Aiming at global analysis, the well known mathematical manifold is perhaps the most important subject of modern mathematics. Based upon mathematical manifold, this numerical manifold method is a newly developed general numerical method. This method computes the movements and deformations of structures or materials. The meshes of the numerical manifold method are finite covers. As the material domains, the finite covers overlapped each other and covered the entire material volume. On each cover, the manifold method defines an independent cover displacement function. The cover displacement functions on individual covers are connected together to form a global displacement function on the entire material volume.

The global displacement function are the weighted averages of local independent cover functions on the common part of several covers. Using the finite cover systems, continuous, jointed or blocky materials can be computed in a mathematically consistent manner. For a manifold computation, the mathematical mesh and physical mesh are independent. Therefore, the mathematical mesh is free to define and free to change. As the mathematical mesh, the covers can be moved, can be split and can be easily removed and added. Moving the covers, the large deformations and moving boundaries can be computed by steps. By dividing a cover to two or more independent covers with their displacement functions, jointed and blocky materials can be modeled.

Both the finite element method (FEM) for continua and the discontinuous deformation analysis (DDA) for block systems are special cases of this numerical manifold method. In the current development stage of numerical manifold method, by using finite cover approach, the extended finite element method can compute more flexible and visible deformations and movements of joints and blocks.

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Chapter 1

General Finite Covers of Manifold Method

1.1 Finite Covers Formed by Mathematical Mesh and Physical Mesh

Based on finite cover systems, the newly developed "manifold method" has the potential to meet more engineering requirements. The term "manifold" here is a generalization of the differential manifold, which is the main subject of differential geometry, topology, differential topology and modern algebra of mathematics. The difference of the "manifold" here and the traditional differential manifold is the following: the global functions of the differential manifold are highly differentiable and entirely defined irrelevant with the covers; the global functions of manifold here defined are based on covers and only piecewise differentiable, mostly discontinuous on the contact interfaces.

Physically, material objects often have different shapes. When the material volumes have fractures, blocks or different zones, the shapes and boundaries become more complex. Under conditions of large deformation and moving boundaries, more difficulty occurs because the conventional analytical approximations are feasible and useful only in a local continuous domain which represents only a small part of whole material volume.

Manifolds connect many individual folded domains together to cover the entire material volume. Then, the global behavior can be computed by functions defined in local covers. The new method has separated and independent mathematical mesh and physical mesh: the mathematical mesh defines only the fine or rough approximations; as the real material boundary, the physical mesh defines the integration fields.

The mathematical mesh is chosen by user, consists of finite overlapping covers which occupy the whole material volume. Conventional meshes and regions, such as regular grids, finite element meshes or convergency regions of series, can be transferred to finite covers of the mathematical mesh. Based on finite covers, manifold method is flexible enough to contain and combine well developed analytical method, widely used FEM method and joint or block oriented DDA method in a unified form.
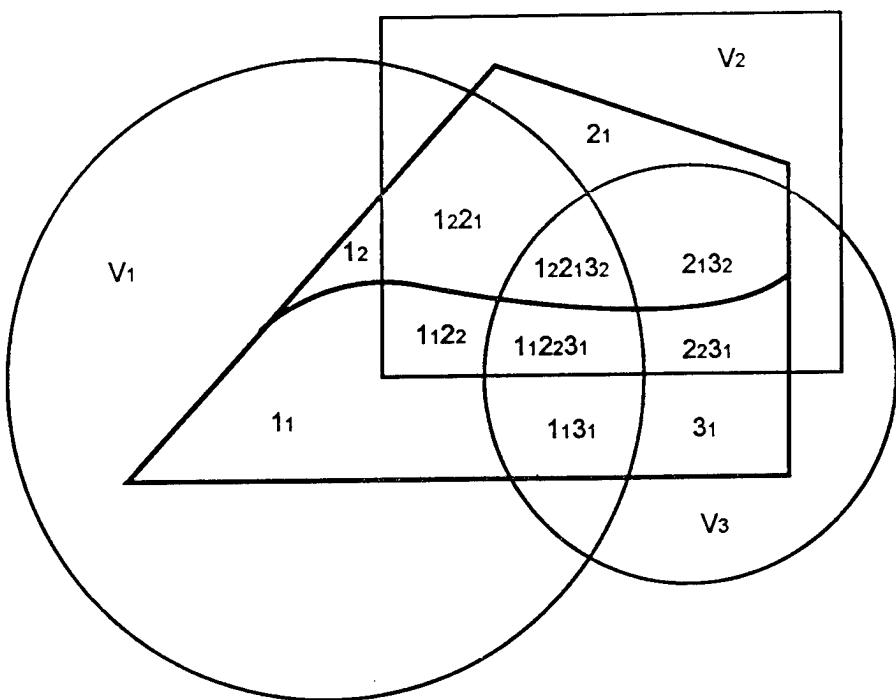
FIGURE 1.1 General covers with one joint
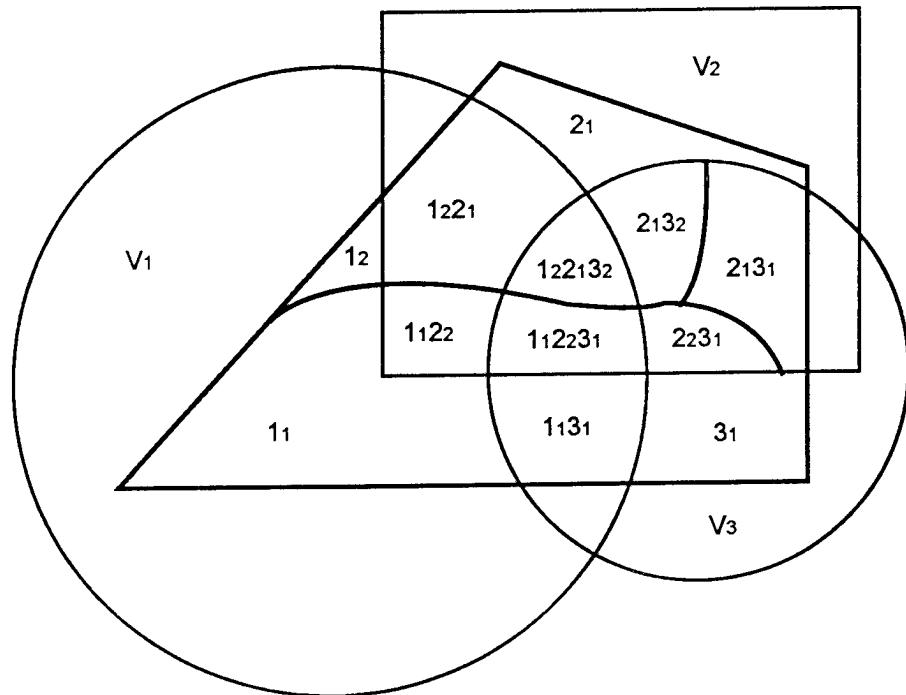
FIGURE 1.2 General covers with two joints

The physical mesh includes the boundaries of the material volume, joints, blocks and the interfaces of different material zones. The constantly changing water surfaces are also part of the physical meshes. The physical mesh represents material conditions which can not be chosen artificially.

The physical cover system is formed by both mathematical and physical meshes. If the joints or block boundaries divide a mathematical cover to two or more completely disconnected domains, those domains are defined as physical covers. Therefore, the physical covers are the subdivision of the mathematical covers by discontinuities. The manifold method is more suitable to compute large deformations, moving boundaries of both continuums and jointed materials.

In Figure 1.1 and 1.2, two circles and one rectangle (indicated by thin lines) delimit three mathematical covers

$$V_1, \ V_2, \ V_3$$

to form the mathematical mesh. The thick lines indicate the material boundary and inner curved joints. In Figure 1.1, V_1 is divided by the physical mesh into two physical covers 1_1, 1_2, V_2 has two physical covers 2_1, 2_2 and V_3 has two physical covers 3_1, 3_2.

Figure 1.2 shows a more complex mesh. Mathematical cover V_2 contains three curved lines, but only two totally disconnected physical covers 2_1, 2_2 are formed. The upper curve (inside cover 2_1) can't cut through rectangle V_2 to form more physical covers, therefore cover 2_1 is a single physical cover. Similarly since mathematical cover V_3 just intersects the end of the upper curves, physical covers 3_1, 3_2 are formed. In both Figure 1.1 and 1.2, the common part of two or more physical covers are defined as "elements" and marked by its cover numbers.

Figure 1.3 shows a simple but often useful chain cover system. This cover system is specially convenient for long and narrow material shapes.

Figure 1.4 shows a DDA block system, here each block is a mathematical cover and a physical cover. There are no overlaps between any two covers in DDA case.

1.2 Cover Functions and Weight Functions on Finite Covers

The normal analytical method or series method work only on very simple domains such as spheres, rectangles and the functions are limited to be highly differentiable. For the
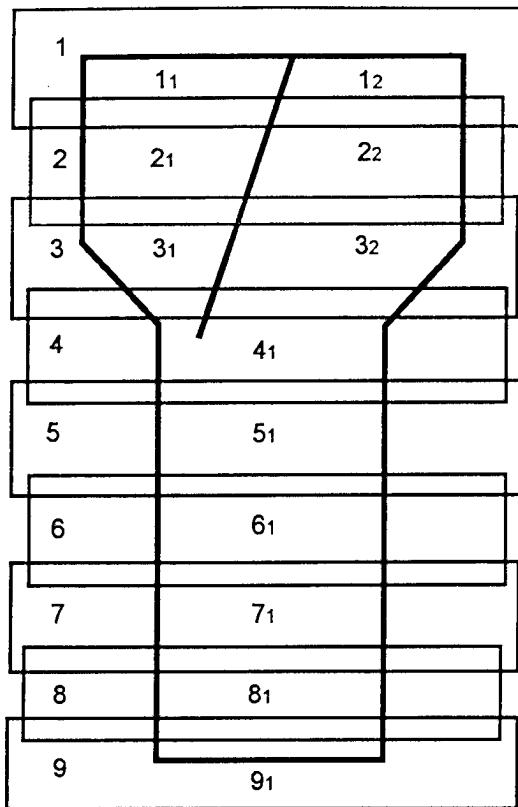
FIGURE 1.3 Chain cover system
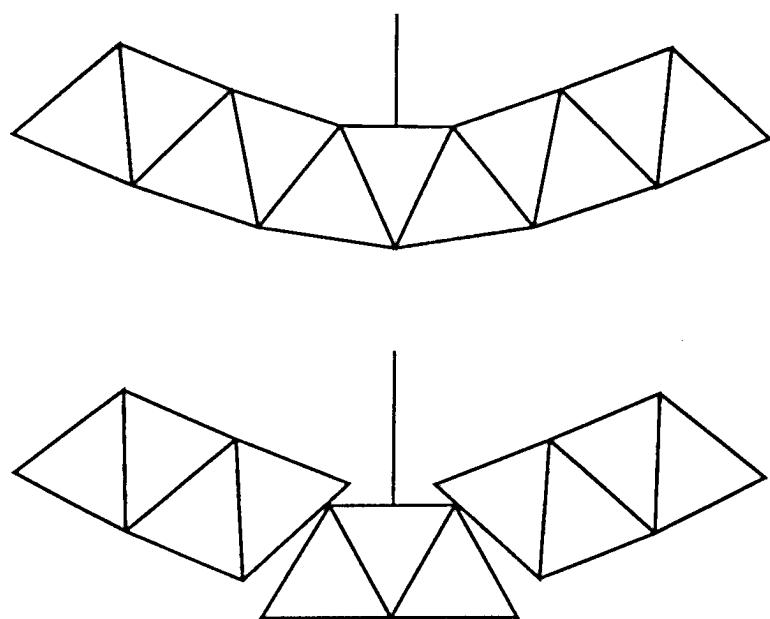
FIGURE 1.4 General covers formed by DDA blocks

manifold method, the cover displacement functions are independently defined on individual physical covers. Local displacement functions can be connected together to form a global displacement function on the whole material volume. Since the joints can cut one cover to more covers as shown in Figure 1.1 and Figure 1.2, the functions are disconnected on the two sides of these joints. The global displacement function is general and flexible enough to represent the wide variety of continuous or discontinuous materials located within moving boundaries.

The cover functions $u_i(x, y)$ defined on physical cover U_i

$$u_i(x, y) \quad (x, y) \in U_i \tag{1.1}$$

can be constant, linear, high order polynomials or locally defined series. These cover functions are connected together by the weight functions $w_i(x, y)$

$$w_i(x, y) \geq 0 \quad (x, y) \in U_i$$
$$w_i(x, y) = 0 \quad (x, y) \notin U_i \tag{1.2}$$

with

$$\sum_{(x,y) \in U_j} w_j(x, y) = 1. \tag{1.3}$$

The meaning of the weight functions $w_i(x, y)$ is weighted average, which is to take a percentage from each cover function $u_i(x, y)$ for all physical covers U_i containing (x, y)

Using the weight functions $w_i(x, y)$ a global function $F(x, y)$ on the whole physical cover system is defined from the cover functions

$$F(x, y) = \sum_{i=1}^{n} w_i(x, y) u_i(x, y). \tag{1.4}$$

Figure 1.5 is a one dimensional example, there are three physical covers

$$U_1 = A_1 A_2, \quad U_2 = B_1 B_2, \quad U_3 = C_1 C_2$$

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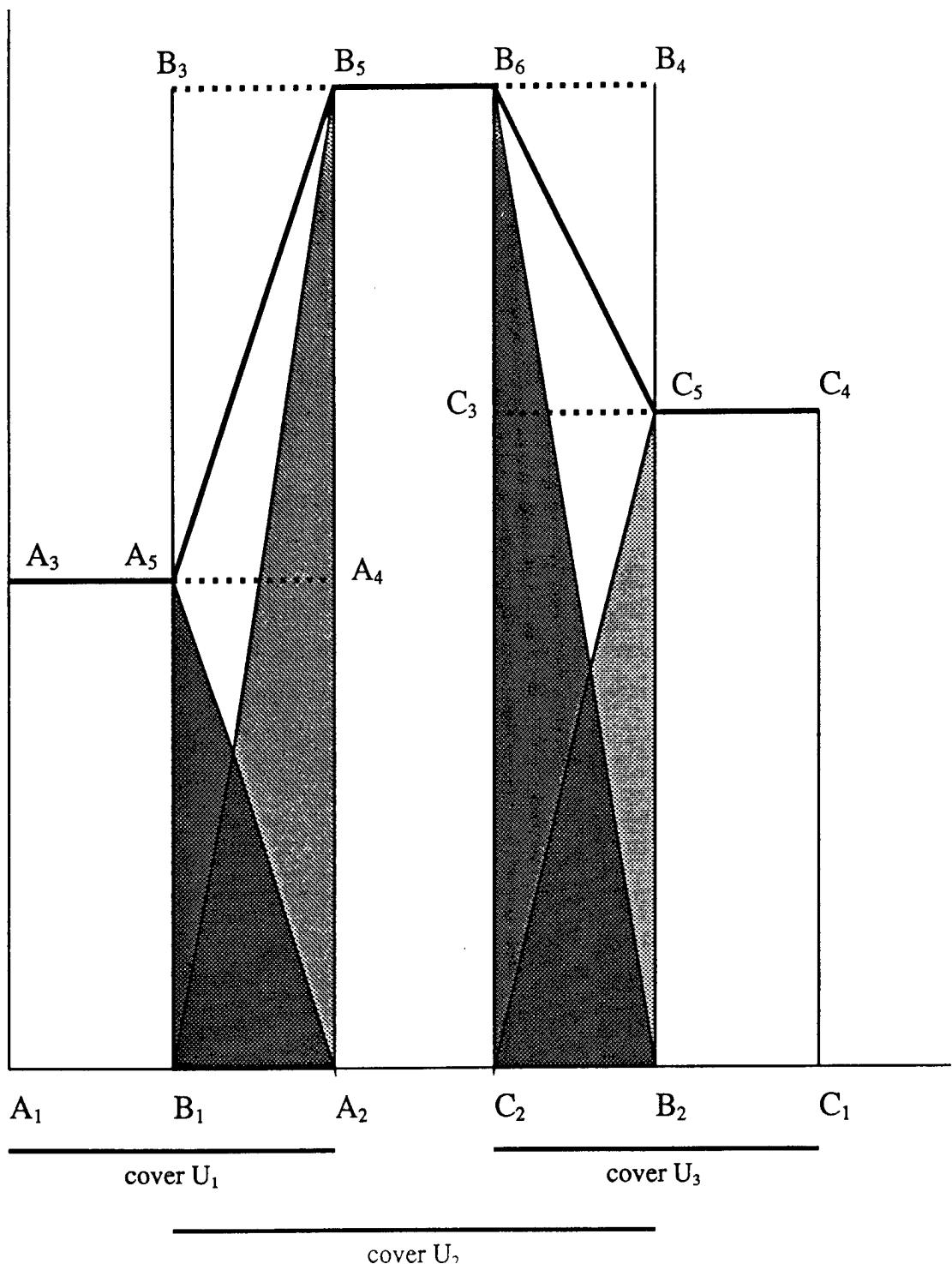
FIGURE 1.5 1-d general covers

$$
\begin{aligned}
u_1(x) &= & A_3 A_4\,, & & x \in U_1 \\
u_2(x) &= & B_3 B_4\,, & & x \in U_2 \\
u_3(x) &= & C_3 C_4\,, & & x \in U_3 \\
w_1(x)u_1(x) &= & A_3 A_5 A_2, & & x \in U_1 \\
w_2(x)u_2(x) &= & B_1 B_5 B_6 B_2, & & x \in U_2 \\
w_3(x)u_3(x) &= & C_2 C_5 C_4, & & x \in U_3
\end{aligned}
$$

The global function $F(x)$ is

$$
F(x) = \sum_{i=1}^{n} w_i(x)u_i(x) = A_3 A_5 B_5 B_6 C_5 C_4.
$$

1.3 Global Functions on Continuous and Discontinuous Materials

For material analysis, four basically different methods are often used. In the order of their development, analytical solutions are the earliest, then came finite difference, the finite element method, and most recently the distinct element method and the discontinuous deformation analysis. The analytic approach has simple material boundary such as rectangles or circles and has no meshes. The finite difference uses grid meshes with equal spaces and as such, is more general than the analytic solution method. The finite element method was a revolution, it shifts from differential equations to integral equations, from the smooth functions to the piece wise smooth functions. The meshes of finite element method can give results of generally shaped continuous materials. The latest distinct element method and discontinuous deformation analysis method are for block systems which are totally discontinuous. The displacement functions of distinct element method and discontinuous deformation analysis are defined for individual blocks of general shape which are completely disconnected from block to block.

As a one dimensional example, Figure 1.6 shows the accuracy of different methods when approaching a natural function (thin curves) which is discontinuous at a point. The thick smooth curve of Figure 1.6 a) is the approximation from the analytical and finite difference methods. The thick piecewise smooth segments of Figure 1.6 b) are the approximation from FEM. The one dimensional elements are

$$
x_0 x_1, x_1 x_2, x_2 x_3, x_3 x_4, x_4 x_5.
$$
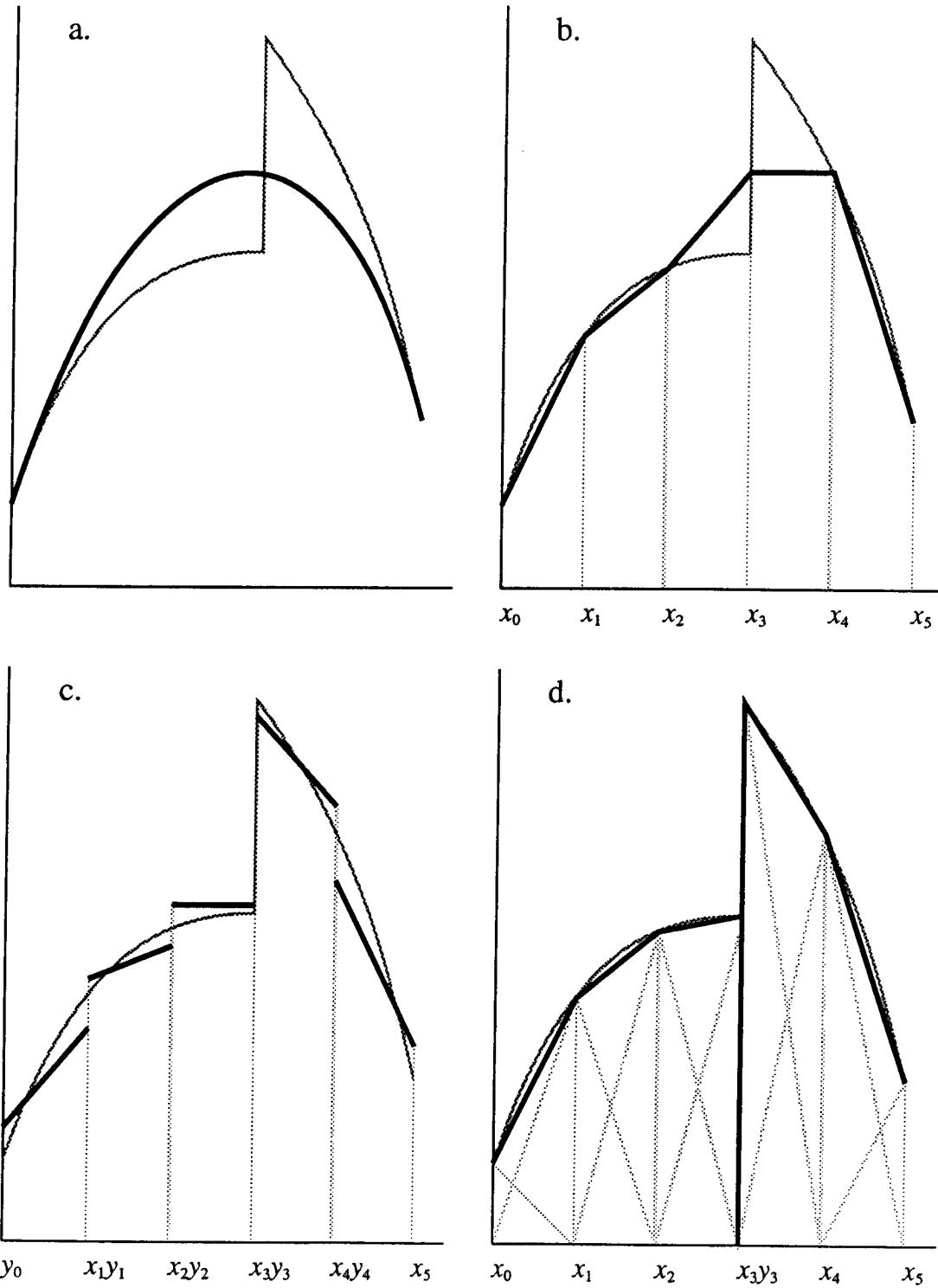
FIGURE 1.6 Accuracy of different methods

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The disconnected segments of Figure 1.6 c) are the approximations from DEM and DDA methods. The one dimensional blocks are

$$y_0 x_1, y_1 x_2, y_2 x_3, y_3 x_4, y_4 x_5.$$

which have more unknowns than the previous methods.

Figure 1.6 d) and Figure 1.7 show the approximation of the manifold method. There are seven one dimensional physical covers

$$U_1 = x_0 x_1, \quad U_2 = x_0 x_2, \quad U_3 = x_1 x_3, \quad U_4 = x_2 x_3,$$

$$U_5 = y_3 x_4, \quad U_6 = y_3 x_5, \quad U_7 = x_4 x_5$$

Since the natural function has a jump at the point $x_3 = y_3$, the mathematical cover $x_2 x_4$ was split to two physical covers $U_4 = x_2 x_3$ and $U_5 = y_3 x_4$.

$$
\begin{aligned}
w_1(x)u_1(x) &= A_0 x_1, & x &\in U_1 \\
w_2(x)u_2(x) &= x_0 A_1 x_2, & x &\in U_2 \\
w_3(x)u_3(x) &= x_1 A_2 x_3, & x &\in U_3 \\
w_4(x)u_4(x) &= x_2 A_3, & x &\in U_4 \\
w_5(x)u_5(x) &= B_3 x_4, & x &\in U_5 \\
w_6(x)u_6(x) &= y_3 A_4 x_5, & x &\in U_6 \\
w_7(x)u_7(x) &= x_4 A_5, & x &\in U_7
\end{aligned}
$$

The global function

$$F(x) = \sum_{i=1}^{7} w_i(x)u_i(x) = A_0 A_1 A_2 A_3 B_3 A_4 A_5 \tag{1.5}$$

is very close to the original natural curve. The global displacement functions of the manifold method are capable of representing large deformations of fractured or blocky materials until the ultimate damage stage in a unified mathematical form.

For the discontinuous deformation analysis, the material body is simply individual blocks. Each block is a mathematical cover, and each mathematical cover is a physical cover. The mathematical mesh and physical mesh are the same where all covers are not overlapped. Therefore the discontinuous deformation analysis is the totally discontinuous case of manifold method.
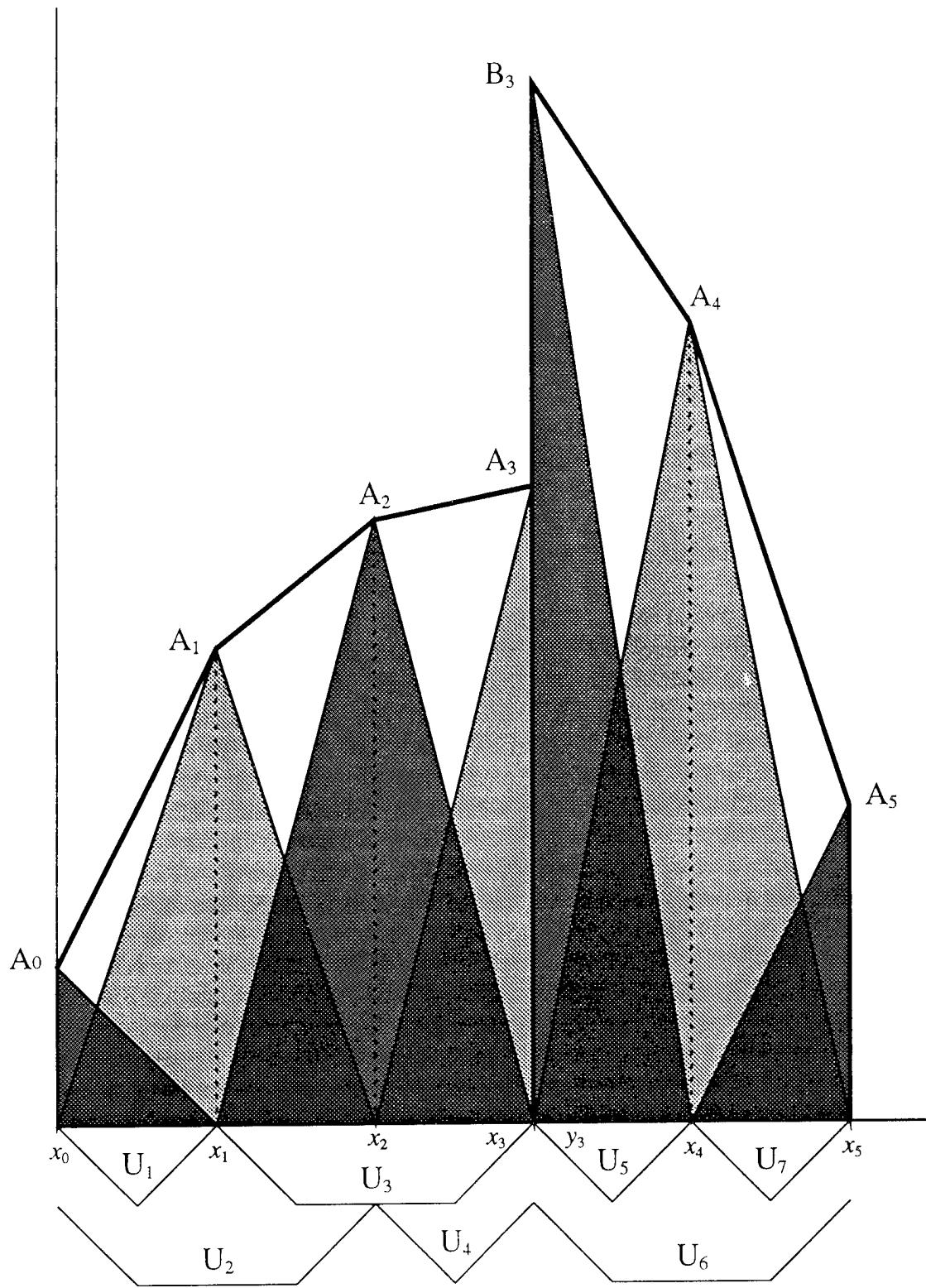
FIGURE 1.7 Accuracy of manifold method

1.4 Elements As The Common Part of Overlapped Covers

For two dimensional manifold computation, the cover displacement functions $u_i(x,y)$ and $v_i(x,y)$ are defined on cover U_i. The global displacement functions $u(x,y)$ and $v(x,y)$ on the whole material body are two global functions:

The cover displacement functions $u_i(x,y)$ and $v_i(x,y)$ defined on physical cover U_i

$$u_i(x,y) \quad (x,y) \in U_i$$

$$v_i(x,y) \quad (x,y) \in U_i$$

can be constant, linear, high order polynomials or locally defined series. These cover displacement functions are connected together by the weight functions $w_i(x,y)$

$$w_i(x,y) \geq 0 \quad (x,y) \in U_i$$

$$w_i(x,y) = 0 \quad (x,y) \notin U_i \quad\quad\quad (1.6)$$

with

$$\sum_{(x,y)\in U_j} w_j(x,y) = 1$$

Using the weight functions $w_i(x,y)$ a global displacement function $u(x,y)$ and $v(x,y)$ on the whole physical cover system is defined from the cover displacement functions

$$u(x,y) = \sum_{i=1}^{n} w_i(x,y)u_i(x,y)$$

$$v(x,y) = \sum_{i=1}^{n} w_i(x,y)v_i(x,y)$$

Each common domain of physical covers is defined as a element e, From formula (1.6), in each element, weight function $w_i(x,y)$ has a analytical representation, which is

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either constant 0 or a differentiable elementary function. Therefore, global displacement function (u, v) has analytical representation in each element.

The cover displacement functions can be given in the following form:

$$
\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}
= [T_i(x,y)]\,[D_i]
$$

$$
= \begin{pmatrix} t_{11}(x,y) & t_{12}(x,y) & t_{13}(x,y) & \ldots & t_{1m-1}(x,y) & t_{1m}(x,y) \\ t_{21}(x,y) & t_{22}(x,y) & t_{23}(x,y) & \ldots & t_{2m-1}(x,y) & t_{2m}(x,y) \end{pmatrix}
\begin{pmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \\ d_{i4} \\ \vdots \\ d_{im-1} \\ d_{im} \end{pmatrix}
$$

$$(1.7)$$

where the subscript "i" represents the i-th cover.

If a element e is not in cover U_i, $T_i(x,y) = 0$.

$$
T_i(x,y) \neq 0 \quad e \in U_i
$$
$$
T_i(x,y) = 0 \quad e \notin U_i
$$

1.5 Possible Displacement Functions for General Covers

On cover U_i, $(u_i(x,y), v_i(x,y))$ is the displacements of point (x,y) on x any y direction respectively, The cover displacement functions $(u_i(x,y), v_i(x,y))$ often take one of the following forms:

the constant function on cover U_i

$$
u_i(x,y) = d_{i1}
$$
$$
v_i(x,y) = d_{i2} \tag{1.8}
$$

the complete first order approximation on cover U_i

$$
u_i(x,y) = d_{i1} + d_{i3}x + d_{i5}y
$$
$$
v_i(x,y) = d_{i2} + d_{i4}x + d_{i6}y \tag{1.9}
$$

the complete second order approximation on cover U_i

$$u_i(x,y) = d_{i1} + d_{i3}x + d_{i5}y + d_{i7}x^2 + d_{i9}xy + d_{i11}y^2$$

$$v_i(x,y) = d_{i2} + d_{i4}x + d_{i6}y + d_{i8}x^2 + d_{i10}xy + d_{i12}y^2 \qquad (1.10)$$

or the general series form $u_i(x,y)$ and $v_i(x,y)$ on cover U_i

$$\left\{ \begin{array}{c} u_i(x,y) \\ v_i(x,y) \end{array} \right\} =$$

$$\begin{pmatrix} f_1(x,y) & 0 & f_2(x,y) & 0 & \cdots & \cdots & f_m(x,y) & 0 \\ 0 & f_1(x,y) & 0 & f_2(x,y) & \cdots & \cdots & 0 & f_m(x,y) \end{pmatrix} \begin{pmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \\ d_{i4} \\ \vdots \\ \vdots \\ d_{i2m-1} \\ d_{i2m} \end{pmatrix}$$

$$(1.11)$$

1.6 Coefficient Matrix of Equilibrium Equations for General Covers

Assume the number of physical covers is n, and there are $2m$ unknowns in each physical cover,

$$D_i = \begin{pmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \\ d_{i4} \\ \vdots \\ d_{i2m-1} \\ d_{i2m} \end{pmatrix} \qquad i = 1, 2, \ldots, n \,.$$

the total potential energy has the form

$$
\Pi = \frac{1}{2} \begin{pmatrix} D_1^T & D_2^T & D_3^T & \ldots & D_n^T \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} & \ldots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \ldots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \ldots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \ldots & K_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \end{pmatrix}
$$

$$
+ \begin{pmatrix} D_1^T & D_2^T & D_3^T & \ldots & D_n^T \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} + C \,. \tag{1.12}
$$

Because each cover has $2m$ degrees of freedom, each submatrix K_{ij} in the coefficient matrix given by equation (1.12) is a $2m \times 2m$ matrix. D_i and F_i are $2m \times 1$ submatrices, where D_i represents the displacement variables $(d_{i1}\ d_{i2}\ d_{i3}\ d_{i4}\ \ldots\ d_{i2m})^T$ of physical cover i. From the formulation of Π, the formula (1.12) can be written as a symmetric representation,

$$
K_{ij} = K_{ji}^T
$$

The equilibrium equations are derived from minimizing the total potential energy Π. The i-th row of following equation (1.13) consists of $2m$ linear equations

$$
\frac{\partial \Pi}{\partial d_{ir}} = 0, \quad r = 1, 2, 3, 4, \ldots, 2m,
$$

where d_{ir} is the displacement variable of cover i. The matrix of obtained simultaneous equilibrium equations is same as the matrix of quadratic form (1.12):

$$
\begin{pmatrix} K_{11} & K_{12} & K_{13} & \ldots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \ldots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \ldots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \ldots & K_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} \,. \tag{1.13}
$$

For material analysis, F_i is the loading on cover i distributed to the $2m$ displacement variables. Submatrices $[K_{ii}]$ depend on the material properties of cover i and $[K_{ij}]$, where $i \neq j$ is defined by the overlapping or contact between cover i and cover j.

Chapter 2

Element Matrices of General Finite Covers

2.1 Stiffness Matrix for General Covers

For the manifold method with general covers, the elements are the common domains intersected by the physical covers. The integration domains of the stiffness matrices are the whole elements which are the intersection of several physical covers.

Same as FEM method, the relationship between stress and strain, is given by

$$
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = [E] \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix},
$$

where

$$
[E] = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix}, \tag{2.1}
$$

and E, ν are Young's modulus and Poisson's ratio respectively. And

$$
\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \sum_{i=1}^{n} [T_i]\{D_i\}
$$

$$
= ([T_1] \quad [T_2] \quad \ldots \quad [T_n]) \begin{pmatrix} \{D_1\} \\ \{D_2\} \\ \{\vdots\} \\ \{D_n\} \end{pmatrix}, \tag{2.2}
$$

$$
\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u(x,y)}{\partial x} \\ \frac{\partial v(x,y)}{\partial y} \\ \frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x} \end{pmatrix}
$$

$$
[T_i(x,y)] =
$$

$$
= \begin{pmatrix} t_{11}(x,y) & t_{12}(x,y) & t_{13}(x,y) & \ldots & t_{1m-1}(x,y) & t_{1m}(x,y) \\ t_{21}(x,y) & t_{22}(x,y) & t_{23}(x,y) & \ldots & t_{2m-1}(x,y) & t_{2m}(x,y) \end{pmatrix}
$$

Then

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \tau_{xy} \end{pmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{m} \begin{pmatrix} \frac{\partial t_{1j}}{\partial x} \\ \frac{\partial t_{2j}}{\partial y} \\ \frac{\partial t_{1j}}{\partial y} + \frac{\partial t_{2j}}{\partial x} \end{pmatrix} \{d_{ij}\} \tag{2.3}$$

For cover i, denote

$$[B_i] == \begin{pmatrix} \frac{\partial t_{11}}{\partial x} & \frac{\partial t_{12}}{\partial x} & \frac{\partial t_{13}}{\partial x} & \cdots & \frac{\partial t_{1m}}{\partial x} \\ \frac{\partial t_{21}}{\partial y} & \frac{\partial t_{22}}{\partial y} & \frac{\partial t_{23}}{\partial y} & \cdots & \frac{\partial t_{2m}}{\partial y} \\ \frac{\partial t_{11}}{\partial y} + \frac{\partial t_{21}}{\partial x} & \frac{\partial t_{12}}{\partial y} + \frac{\partial t_{22}}{\partial x} & \frac{\partial t_{13}}{\partial y} + \frac{\partial t_{23}}{\partial x} & \cdots & \frac{\partial t_{1m}}{\partial y} + \frac{\partial t_{2m}}{\partial x} \end{pmatrix} \tag{2.4}$$

Then

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} [B_1] & [B_2] & [B_3] & \vdots & [B_n] \end{pmatrix} \begin{pmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \\ \vdots \\ \{D_n\} \end{pmatrix} = [B][D] \tag{2.5}$$

where

$$[B] = \begin{pmatrix} [B_1] & [B_2] & [B_3] & \vdots & [B_n] \end{pmatrix}$$

$$[D] = \begin{pmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \\ \vdots \\ \{D_n\} \end{pmatrix}$$

The strain energy Π_e done by the elastic stresses of element e is

$$\Pi_e = \iint_{\mathcal{A}} \frac{1}{2} (\epsilon_x \sigma_x + \epsilon_y \sigma_y + \gamma_{xy} \tau_{xy}) \, dx \, dy, \tag{2.6}$$

where the integration is over the entire material area $\mathcal{A}$ in that element. Then

$$\Pi_e = \iint_{\mathcal{A}} \frac{1}{2} \begin{pmatrix} \epsilon_x & \epsilon_y & \gamma_{xy} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} \, dx \, dy$$

$$= \frac{1}{2} \iint_A \{D\}^T [B]^T [E][B]\{D\} \, dx \, dy$$

$$= \frac{1}{2} \{D\}^T \left[\iint_A [B]^T [E][B] \, dx \, dy \right] \{D\}$$

$$= \frac{1}{2} \{D\}^T \left(S^e [B]^T [E][B] \right) \{D\}, \tag{2.7}$$

where S^e is the area of that element.

Therefore,

$$S^e [B]^T [E][B] = S^e \begin{pmatrix} [B_1]^T \\ [B_2]^T \\ [B_3]^T \\ \vdots \\ [B_n]^T \end{pmatrix} [E] \Big([B_1] \quad [B_2] \quad [B_3] \quad \cdots \quad [B_n] \Big), \tag{2.8}$$

is the element stiffness matrix.

Then

$$S^e [B_r]^T [E][B_s] \qquad \rightarrow \qquad [K_{rs}], \qquad r, s = 1, 2, 3, \ldots, n.$$

2.2 Initial Stress Matrix for General Covers

Following the time sequence, the manifold method computes step by step. The computed stresses of previous time step will be the initial stresses of the next time step. Therefore the initial stresses are essential for manifold computation.

For the element e, the potential energy of the initial constant stresses $\{\sigma_e^0\} = \left(\sigma_x^0 \quad \sigma_y^0 \quad \tau_{xy}^0 \right)^T$ is

$$\Pi_\sigma = \iint_A (\epsilon_x \sigma_x^0 + \epsilon_y \sigma_y^0 + \gamma_{xy} \tau_{xy}^0) \, dx \, dy$$

$$= \iint_A \left(\epsilon_x \quad \epsilon_y \quad \gamma_{xy} \right) \begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} \, dx \, dy$$

$$= \iint_A \{D\}^T [B]^T \{\sigma_e^0\} \, dx \, dy$$

$$= S^e \{D\}^T [B]^T \{\sigma_e^0\}, \tag{2.9}$$

where S^e is the area of that element. Therefore,

$$-S^e[B_e]^T\{\sigma_e^0\} = -S^e \begin{pmatrix} [B_1]^T \\ [B_2]^T \\ [B_3]^T \\ \vdots \\ [B_n]^T \end{pmatrix} \begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} \tag{2.10}$$

is the load matrix.

Then

$$-S^e[B_r]^T \begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} \qquad \rightarrow \qquad \{F_r\}, \qquad r = 1, 2, 3, \ldots, n.$$

2.3 Point Loading Matrix for General Covers

Different from ordinary FEM method, a load point can be any point in its element. The point loading force $(\ F_x \quad F_y\ )^T$ acts on point (x, y) of element e. And the displacements on force point (x, y) is

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

The potential energy due to the point loading is

$$\begin{aligned} \Pi_p &= -(F_x u(x, y) + F_y v(x, y)) \\ &= -(\ u(x, y) \quad v(x, y)\ ) \begin{pmatrix} F_x \\ F_y \end{pmatrix} \\ &= -\{D\}^T [T(x, y)]^T \begin{pmatrix} F_x \\ F_y \end{pmatrix}. \end{aligned} \tag{2.11}$$

Therefore,

$$[T]^T \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} [T_1]^T \\ [T_2]^T \\ [T_3]^T \\ \vdots \\ [T_n]^T \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} \tag{2.12}$$

is the load matrix.

Then

$$[T_r]^T \begin{pmatrix} F_x \\ F_y \end{pmatrix} \quad \rightarrow \quad \{F_r\}, \qquad r = 1, 2, 3, \ldots, n.$$

2.4 Body Loading Matrix for General Covers

Assuming that $\begin{pmatrix} f_x & f_y \end{pmatrix}^T$ is the constant body force loading acting on the material area of element e.

The potential energy due to the body loading is

$$
\begin{aligned}
\Pi_w &= - \iint_{\mathcal{A}} \left(f_x u(x,y) + f_y v(x,y) \right) dx\, dy \\
&= - \iint_{\mathcal{A}} \begin{pmatrix} u(x,y) & v(x,y) \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} dx\, dy \\
&= -\{D\}^T \left[\iint_{\mathcal{A}} [T]^T \, dx\, dy \right] \begin{pmatrix} f_x \\ f_y \end{pmatrix}.
\end{aligned}
\tag{2.13}
$$

Therefore,

$$
\left[\iint_{\mathcal{A}} [T]^T \, dx\, dy \right] \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \left[\iint_{\mathcal{A}} \begin{pmatrix} [T_1(x,y)]^T \\ [T_2(x,y)]^T \\ [T_3(x,y)]^T \\ \vdots \\ [T_n(x,y)]^T \end{pmatrix} dx\, dy \right] \begin{pmatrix} f_x \\ f_y \end{pmatrix}
$$

$$
= \begin{pmatrix} \iint_{\mathcal{A}} [T_1(x,y)]^T \, dx\, dy \\ \iint_{\mathcal{A}} [T_2(x,y)]^T \, dx\, dy \\ \iint_{\mathcal{A}} [T_3(x,y)]^T \, dx\, dy \\ \vdots \\ \iint_{\mathcal{A}} [T_n(x,y)]^T \, dx\, dy \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix}.
\tag{2.14}
$$

and

$$
\begin{pmatrix} \iint_{\mathcal{A}} [T_1(x,y)]^T \, dx\, dy \\ \iint_{\mathcal{A}} [T_2(x,y)]^T \, dx\, dy \\ \iint_{\mathcal{A}} [T_3(x,y)]^T \, dx\, dy \\ \vdots \\ \iint_{\mathcal{A}} [T_n(x,y)]^T \, dx\, dy \end{pmatrix}
$$

is the body loading matrix. And

Then

$$\left[\iint_{\mathcal{A}} [T_r(x,y)]^T \, dx \, dy \right] \begin{pmatrix} f_x \\ f_y \end{pmatrix} \quad \rightarrow \quad \{F_r\}, \qquad r = 1, 2, 3, \dots, n.$$

2.5 Inertia Force Matrix for General Covers

Inertia force matrix is equivalent to mass matrix of FEM. This matrix is the most important matrix of manifold method. Giving a small time steps, inertia force matrix will control the movements and the stability of all points of the whole material volume. In each time step, the displacements should be small enough to have the final result independent from the choosing of the specific time steps.

Considering the current time step, denote $(\, u(x,y,t) \quad v(x,y,t) \,)^T$ as the time dependent displacements of any point (x,y) of element e and $\mathcal{M}$ as the mass per unit area. The force of inertia per unit area is

$$\begin{pmatrix} f_x(x,y,t) \\ f_y(x,y,t) \end{pmatrix} = -\mathcal{M}\frac{\partial^2}{\partial t^2} \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \end{pmatrix} = -\mathcal{M}[T]\frac{\partial^2 \{D(t)\}}{\partial t^2}. \qquad (2.15)$$

The potential energy of the inertia force of element e is

$$\begin{aligned} \Pi_i &= - \iint_{\mathcal{A}} (\, u(x,y,t) \quad v(x,y,t) \,) \begin{pmatrix} f_x(x,y,t) \\ f_y(x,y,t) \end{pmatrix} \, dx \, dy \\ &= \iint_{\mathcal{A}} \mathcal{M} \, (\, u(x,y,t) \quad v(x,y,t) \,) \, [T]\frac{\partial^2 \{D(t)\}}{\partial t^2} \, dx \, dy. \end{aligned} \qquad (2.16)$$

Assume $\{D(0)\} = \{0\}$ as the element displacements at the beginning of the time step, $\{D(\Delta)\} = \{D\}$ as the displacements at the end of the time step, Δ as the time interval of this time step. Then

$$\begin{aligned} \{D\} = \{D(\Delta)\} &= \{D(0)\} + \Delta\frac{\partial\{D(0)\}}{\partial t} + \frac{\Delta^2}{2}\frac{\partial^2\{D(0)\}}{\partial t^2} \\ &= \Delta\frac{\partial\{D(0)\}}{\partial t} + \frac{\Delta^2}{2}\frac{\partial^2\{D(0)\}}{\partial t^2}, \end{aligned} \qquad (2.17)$$

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$$\frac{\partial^2 \{D(0)\}}{\partial t^2} = \frac{2}{\Delta^2}\{D\} - \frac{2}{\Delta}\frac{\partial\{D(0)\}}{\partial t} = \frac{2}{\Delta^2}\{D\} - \frac{2}{\Delta}\{V(0)\}, \qquad (2.18)$$

where

$$\{V(0)\} = \frac{\partial\{D(0)\}}{\partial t}, \qquad (2.19)$$

is the velocity at the beginning of the time step. Then the potential energy becomes

$$\Pi_i = \mathcal{M}\{D(\Delta)\}^T \left[\iint_{\mathcal{A}} [T(x,y)]^T [T(x,y)]\, dx\, dy\right] \left(\frac{2}{\Delta^2}\{D(\Delta)\} - \frac{2}{\Delta}\{V(0)\}\right). \qquad (2.20)$$

Therefore,

$$-\left[\iint_{\mathcal{A}} [T(x,y)]^T [T(x,y)]\, dx\, dy\right] \left(\frac{2\mathcal{M}}{\Delta^2}\{D(\Delta)\} - \frac{2\mathcal{M}}{\Delta}\{V(0)\}\right). \qquad (2.21)$$

is the load matrix. Thus,

$$\frac{2\mathcal{M}}{\Delta^2}\left[\iint_{\mathcal{A}} [T(x,y)]^T [T(x,y)]\, dx\, dy\right]$$

$$= \frac{2\mathcal{M}}{\Delta^2}\left[\iint_{\mathcal{A}} \begin{pmatrix} [T_1]^T \\ [T_2]^T \\ [T_3]^T \\ \vdots \\ [T_n]^T \end{pmatrix} \left([T_1] \quad [T_2] \quad [T_3] \quad \ldots \quad [T_n]\right) dx\, dy\right] \qquad \rightarrow \qquad [K],$$

$$(2.22)$$

and

$$\frac{2\mathcal{M}}{\Delta}\left[\iint_{\mathcal{A}} [T(x,y)]^T [T(x,y)]\, dx\, dy\right]\{V_e(0)\}$$

$$= \frac{2\mathcal{M}}{\Delta}\left[\iint_{\mathcal{A}} \begin{pmatrix} [T_1]^T \\ [T_2]^T \\ [T_3]^T \\ \vdots \\ [T_n]^T \end{pmatrix} \left([T_1] \quad [T_2] \quad [T_3] \quad \ldots \quad [T_n]\right) dx\, dy\right]\{V(0)\} \qquad \rightarrow \qquad \{F\}.$$

$$(2.23)$$

Then

$$\frac{2\mathcal{M}}{\Delta^2} \left[\iint_{\mathcal{A}} [T_r(x,y)]^T [T_s(x,y)] \, dx \, dy \right] \quad \rightarrow \quad [K_{rs}], \qquad (2.24)$$

$$r, s = 1, 2, 3, \ldots, n.$$

$$\frac{2\mathcal{M}}{\Delta} \left[\iint_{\mathcal{A}} [T_r(x,y)]^T [T_s(x,y)] \, dx \, dy \right] \{V_s(0)\} \quad \rightarrow \quad \{F_r\}$$

$$r, s = 1, 2, 3, \ldots, n. \qquad (2.25)$$

where

$$\{V_s(0)\} = \frac{\partial}{\partial t} \begin{pmatrix} u_s(0) \\ v_s(0) \end{pmatrix}.$$

2.6 Fixed Point Matrix for General Covers

As a boundary condition, some of the elements are fixed at specific points. The constraint can be applied by using two very stiff springs. Assume the fixed point is (x, y) at element e with the displacements

$$\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

There are two springs which are along the x and y directions respectively. The stiffness of the springs is p. The spring forces are

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} -pu(x,y) \\ -pv(x,y) \end{pmatrix}.$$

The strain energy of the spring is Π_f, then

$$\begin{aligned}
\Pi_f &= \frac{p}{2}(u(x,y)^2 + v(x,y)^2) \\
&= \frac{p}{2} \begin{pmatrix} u(x,y) & v(x,y) \end{pmatrix} \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \\
&= \frac{p}{2} \{D\}^T [T(x,y)]^T [T(x,y)] \{D\}. \qquad (2.26)
\end{aligned}$$

Then

$$p[T(x,y)]^T[T(x,y)]$$

$$=p \begin{pmatrix} [T_1(x,y)]^T \\ [T_2(x,y)]^T \\ [T_3(x,y)]^T \\ \vdots \\ [T_n(x,y)]^T \end{pmatrix} \left([T_1(x,y)] \quad [T_2(x,y)] \quad [T_3(x,y)] \quad \ldots \quad [T_n(x,y)] \right).$$

(2.27)

is the $[K]$ matrix. Therefore,

$$p[T_r(x,y)]^T[T_s(x,y)] \qquad \rightarrow \qquad [K_{rs}], \qquad r,s = 1,2,3,\ldots,n.$$

Chapter 3

Entrance Theory on Contacts for General Covers

3.1 Definition of Contacts for General Covers

Thus far, only individual covers and elements were considered. It is necessary to connect the individual discontinuous boundaries into a system. For the movements of discontinuous boundaries, no tension and no penetration must be satisfied between two contact sides. The following theory of kinematics for material is based on contacts. Finding the contacts in each time step, allying stiff springs on contacts, the discontinuous displacements can be computed.

The time steps can be chosen small enough so that the displacements of all points in the whole material body are less than a pre-defined limit ρ. Therefore, the displacements can be small enough so that the displacements $(u(x,y), v(x,y))$, the rotations $r(x,y)$, the deformations $(\epsilon_x \ \epsilon_y \ \gamma_{xy})$ can be accurately represented as linear functions of the cover unknowns $[D_i]$

$$(u(x,y), v(x,y)) = \sum_{i=1}^{n} [T_i(x,y)]\{D_i\}$$

$$r(x,y) = \sum_{i=1}^{n} [R_i(x,y)]\{D_i\}$$

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = \sum_{i=1}^{n} [B_i(x,y)]\{D_i\} \tag{3.1}$$

Based on the small step displacements, the contacts are defined in the beginning of each time step. Each contacts are formed with two sides. All of the pair of two sides that are possible to contact, penetrate or entrance from one to another at the end of the time step are defined as contacts. Since practically there are no penetrations on the two sides that allowed on the contacts, so the contacts are merely the entrance positions.

There are two kinds of contacts: angle to edge and angle to angle. The edge to edge contacts can be transferred to angle to edge contact. (See Figure 3.1) Assume the maximum step edge rotation is δ, there are the criteria for the contacts:

[1] for angle to edge contact, the minimum distance of the angle vertex to the edge of the contacts is less than 2ρ;

[2] for angle to angle contact, the minimum distance of two angle vertices of the contacts is less than 2ρ; (See Figure 3.1)

[3] for angle to edge contact, when the angle vertex translates to the edge without rotation, the maximum overlapping angle of the angle and the edge is less than 2δ;

[4] for angle to angle contact, when the angle vertex translates to the vertex of other angle without rotation, the maximum overlapping angle of the two angles is less than 2δ; (See Figure 3.2)

The computer program sets the maximum step edge rotation $\delta = 1.5$ degrees.

Figure 3.2 shows two complex blocks with many edges and angles, under this contact criteria, there are only two contacts even if the distance limit 2ρ is lager than block diameters.

The common sense requirements of no penetration and no tension indeed are inequalities. Coloumb's friction law and limited tension are also inequalities. From formula (3.1), the inequalities can be simplified to linear inequalities of $[D_i]$. There are three kinds of linear inequalities:

[1] no penetrations in contacts,

[2] tension force less than tension strength in contacts,

[3] Coloumb's friction law in contacts.

In the computation, an angle to angle contact will be transferred to one or two angle to edge contacts. Based on the orientation, the following formula is used to judge the angle to edge contact penetration.
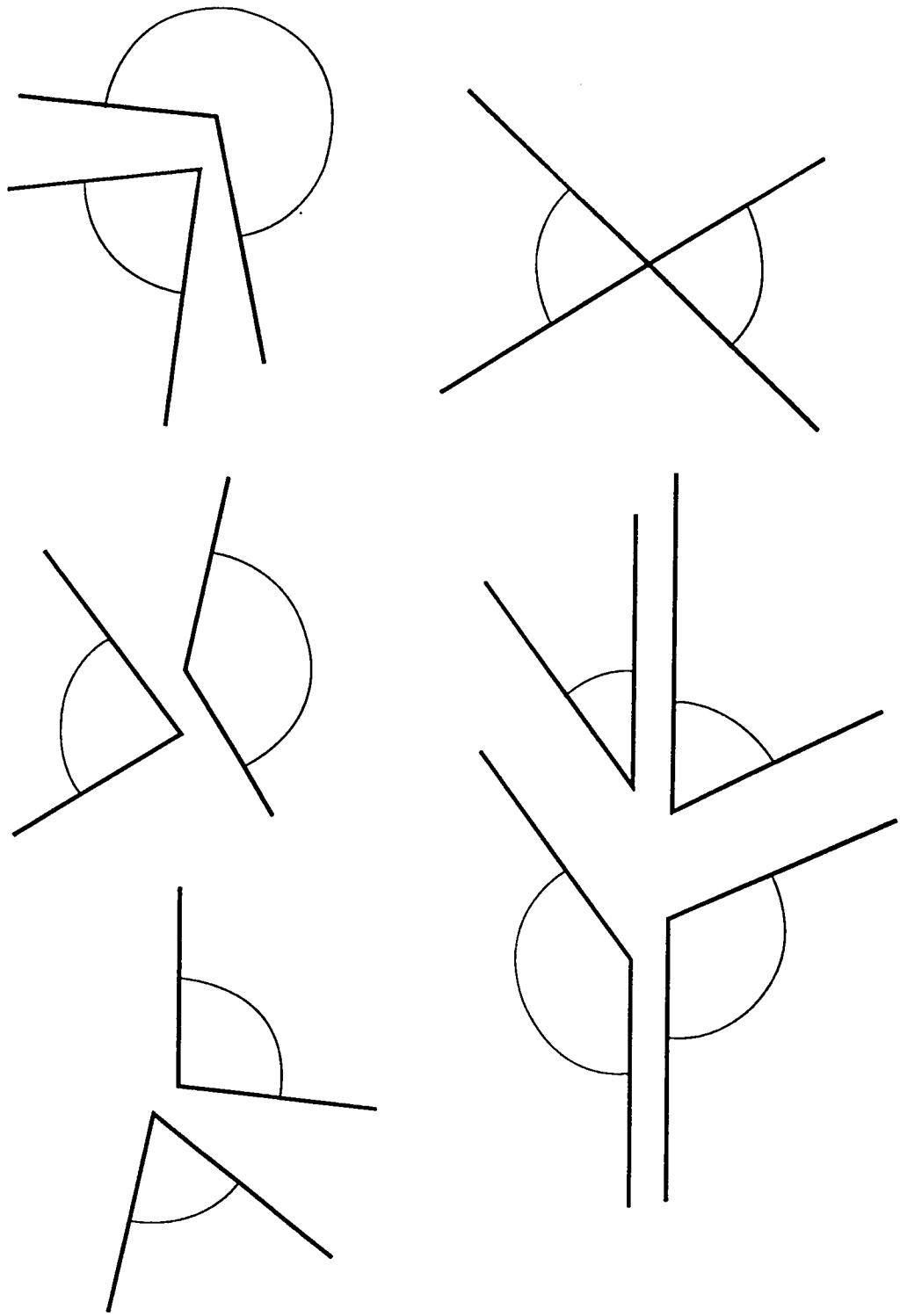
FIGURE 3.1 Two kinds of contacts
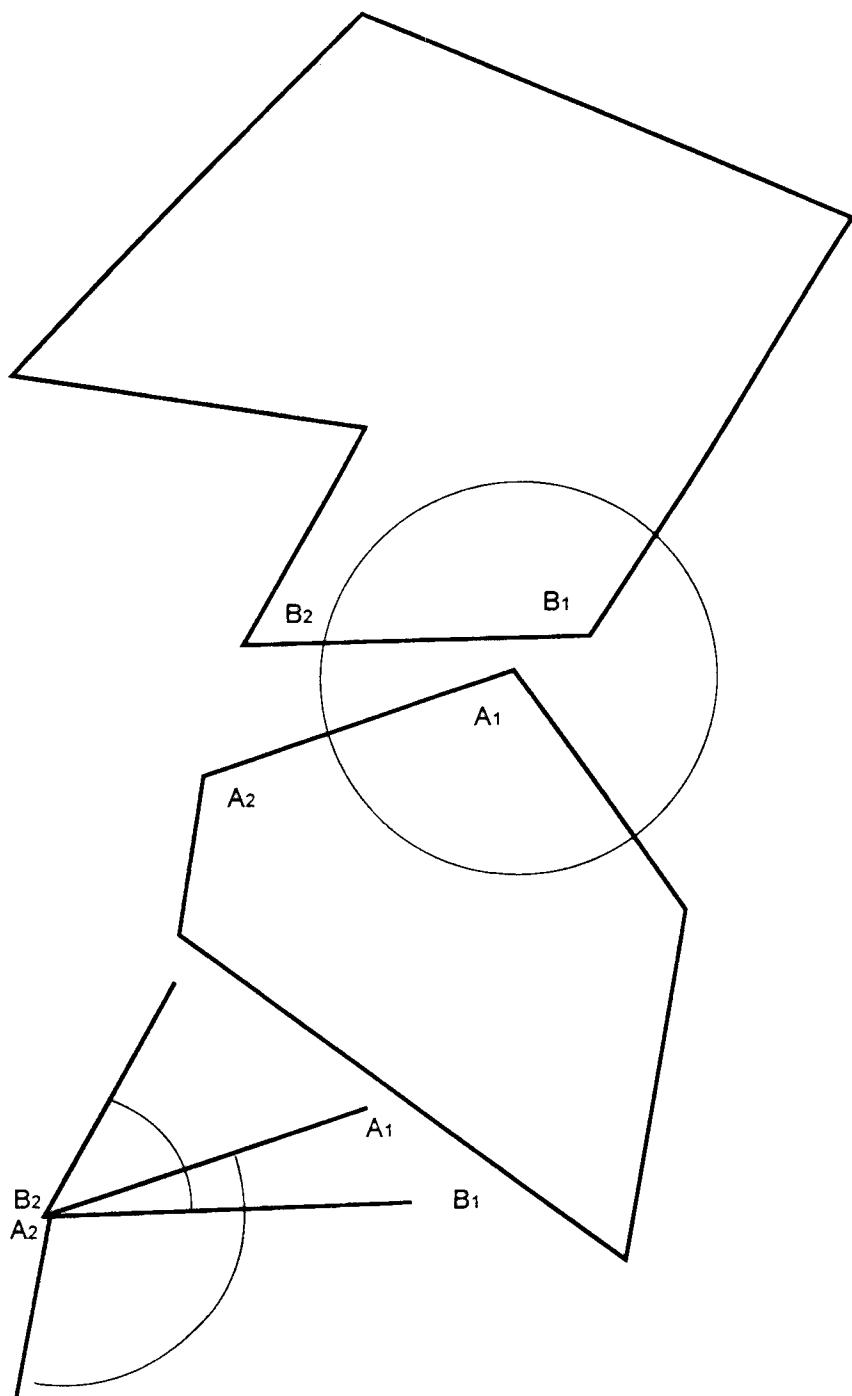
**FIGURE 3.2** contacts between two blocks
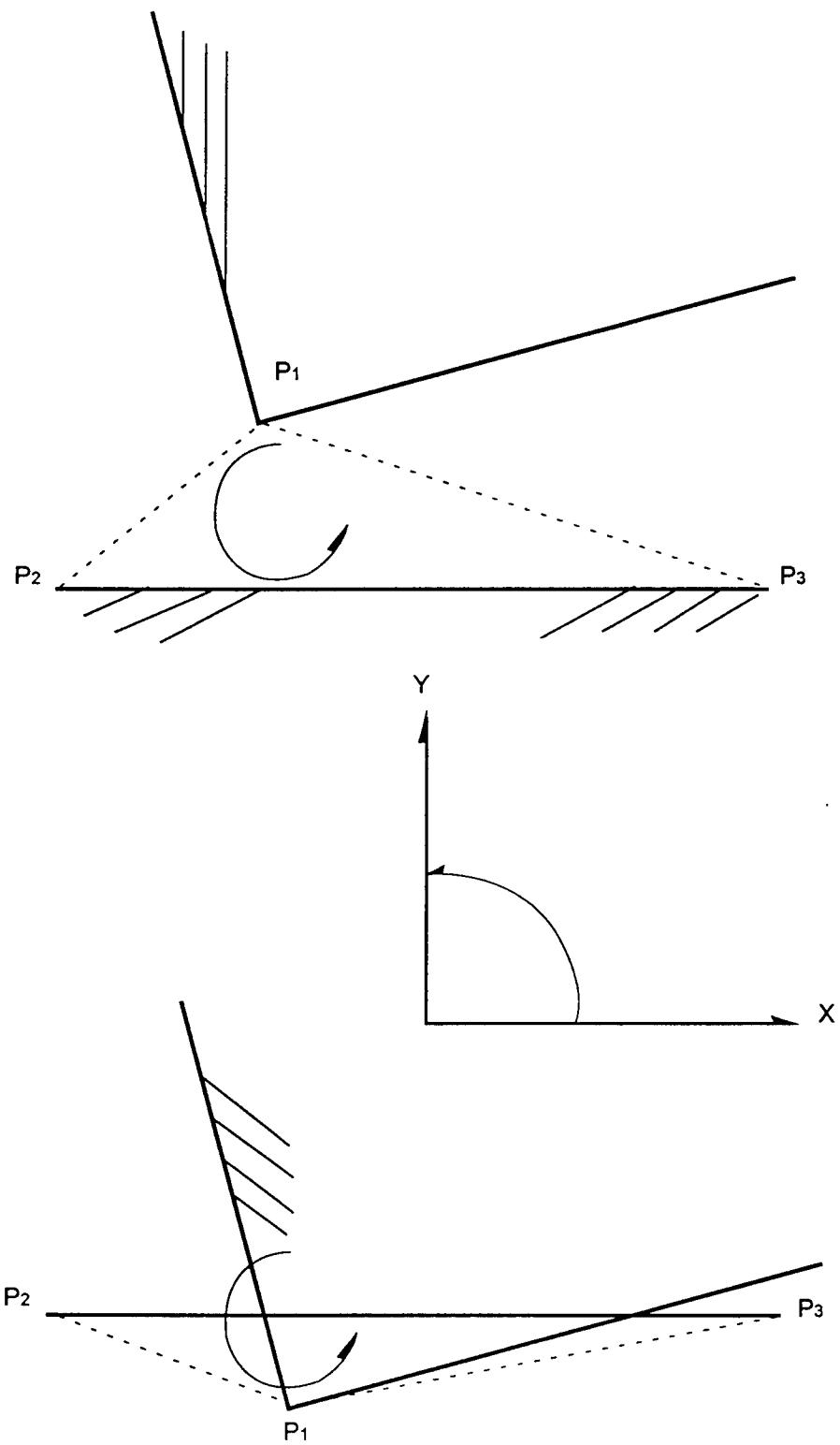
FIGURE 3.3 Judge penetration by rotation
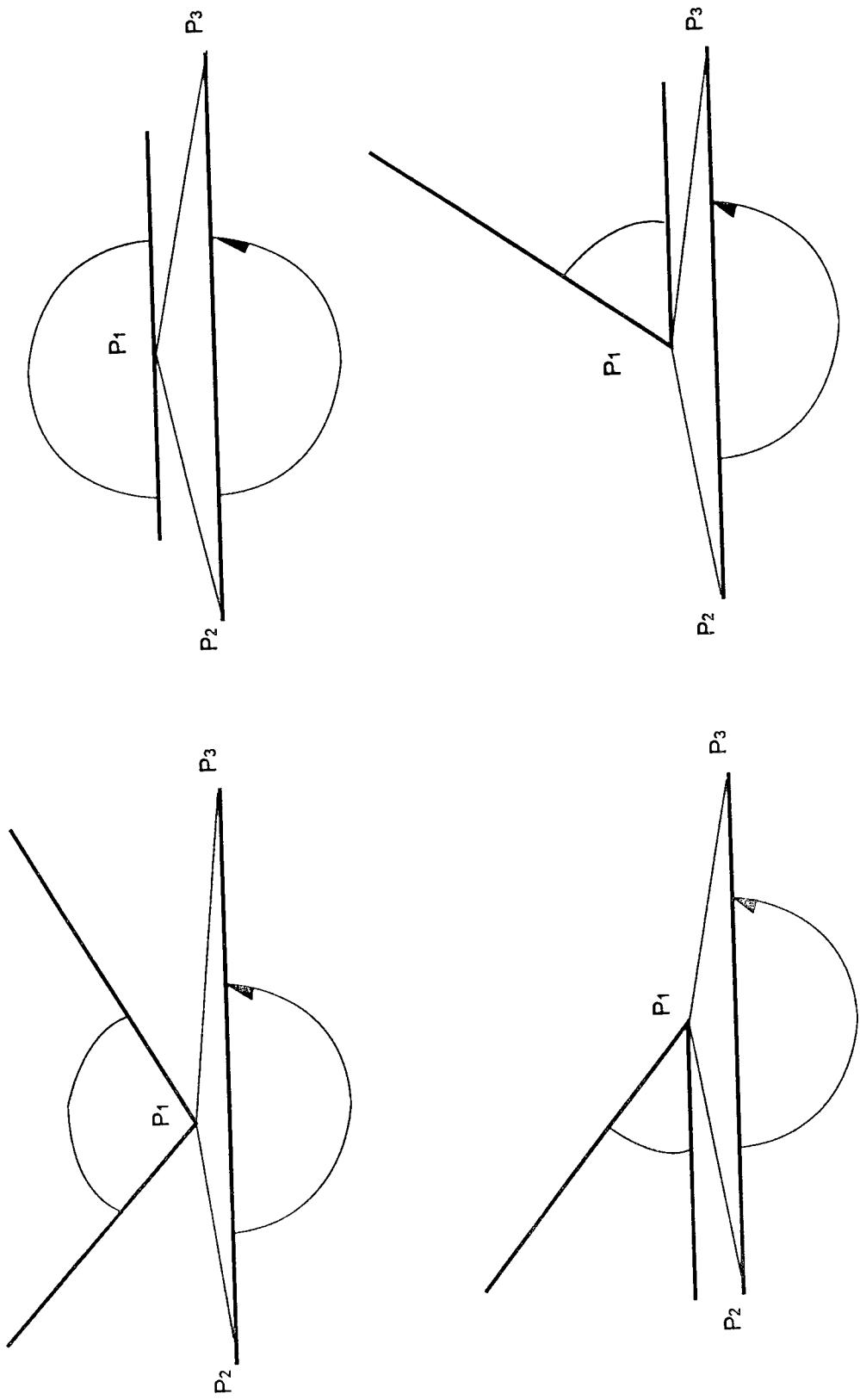
FIGURE 3.4 Judge penetration by rotation

Assume P_1 is a point before deformation which moves to point P_1' after deformation; P_2P_3 is the entrance line and (x_i, y_i) and (u_i, v_i) are the coordinates and displacement increments of $P_i, i = 1, 2, 3$ respectively. If points P_1, P_2 and P_3 rotate in the same sense as the rotation of ox to oy (see Figure 3.3 and Figure 3.4), then P_1' has passed line P_2P_3 and is stated by the inequality:

$$\Delta = \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} < 0. \tag{3.2}$$

This simple formula is still correct even when these three points move simultaneously.

3.2 Entrance Lines of Contacts for General Covers

The kinematics can be imposed on the global equations by adding stiff springs to lock the movement in one or two directions. The theory of block and joint kinematics will decide where and when to put the stiff springs so that the movements of blocks and joints are basically the same as in the real space. For the kinematics, the formulae of penetration judge and penetration lock have to be consistent to ensue the convergency of open-close iterations.

For the contacts, the "entrance" lines can be defined. A contact between two convex angles is shown by Figure 3.5, where the two thick lines are the entrance lines. Penetration will occur if the two entrance lines are passed by the vertices of the other angles simultaneously.

[1] For the angle to angle contact, if both angles are less than $180°$, the two entrance lines are defined according to the following table:

		two entrance lines	
$\alpha \le 180°$	$\beta \le 180°$	OE_3	OE_2
$\alpha \le 180°$	$\beta > 180°$	OE_3	OE_4
$\alpha > 180°$	$\beta \le 180°$	OE_1	OE_2
$\alpha > 180°$	$\beta > 180°$	OE_1	OE_4

[2] For the angle to angle contact, if a angle is larger than $180°$, the two entrance lines are the two edges of the angle greater than $180°$.
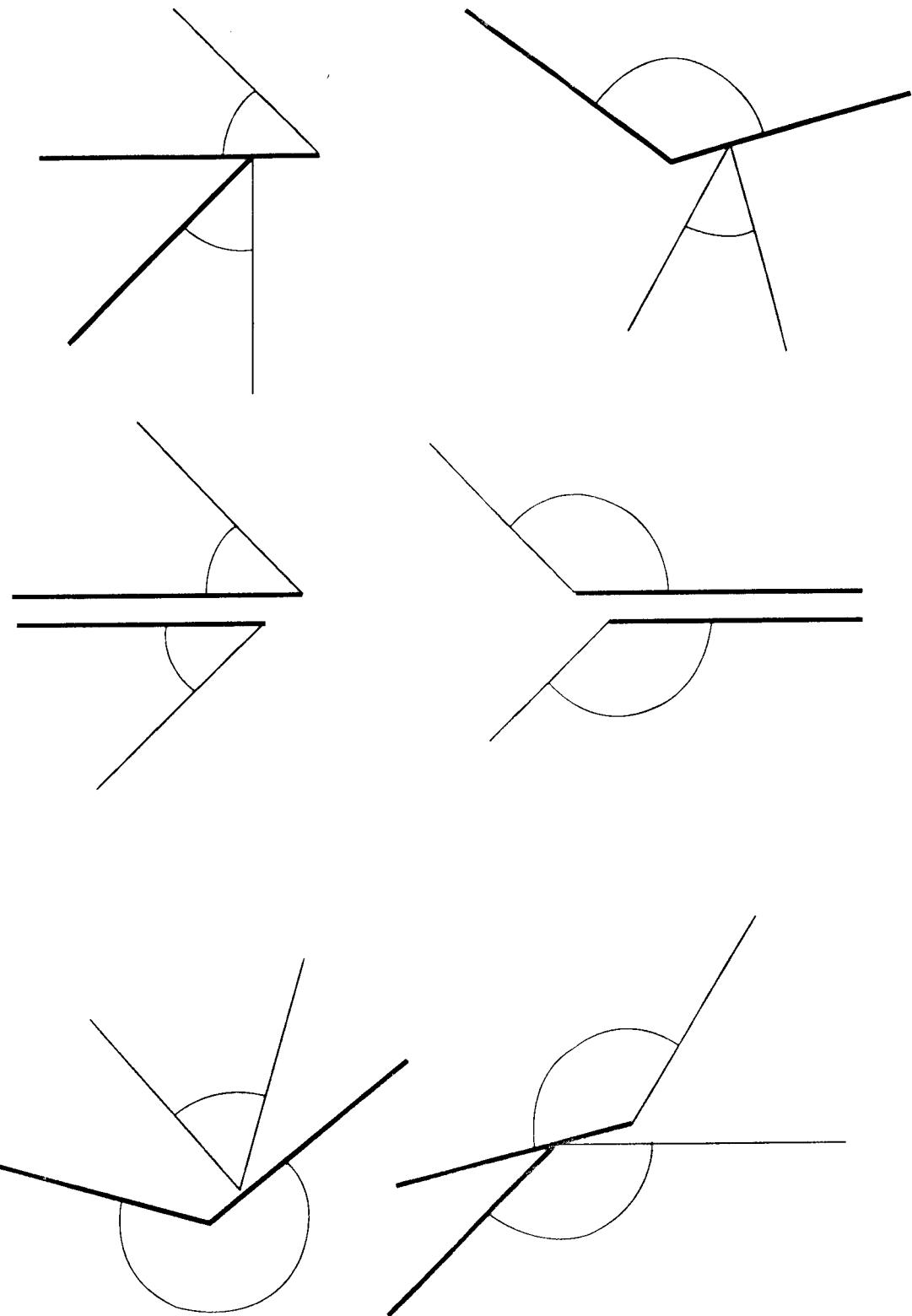
FIGURE 3.5a Entrance lines of different contacts
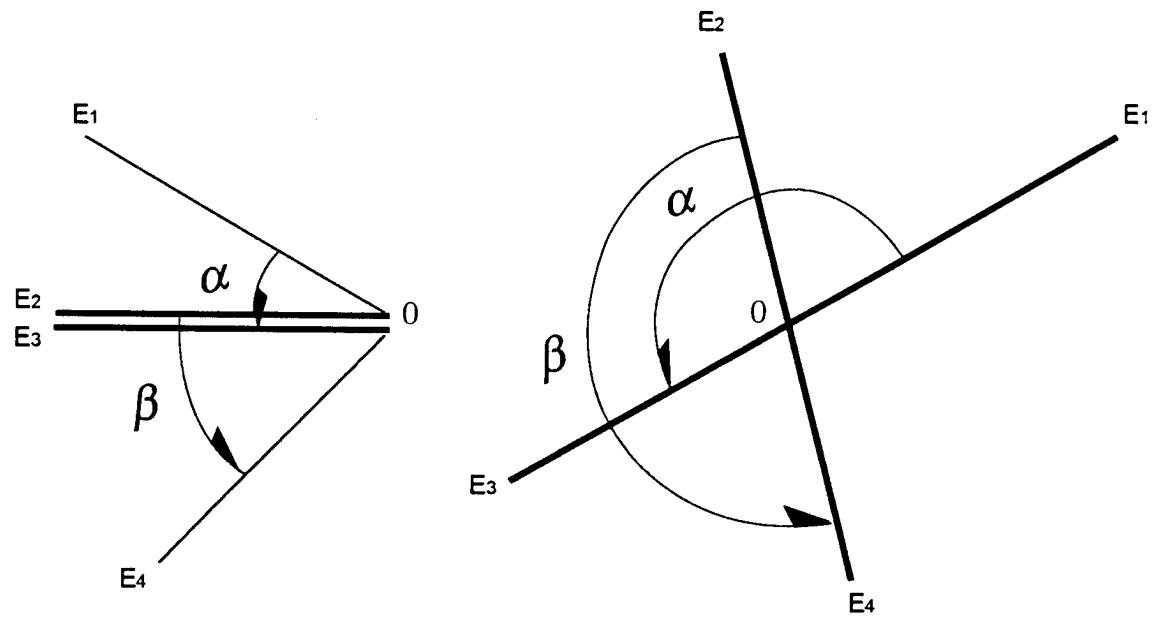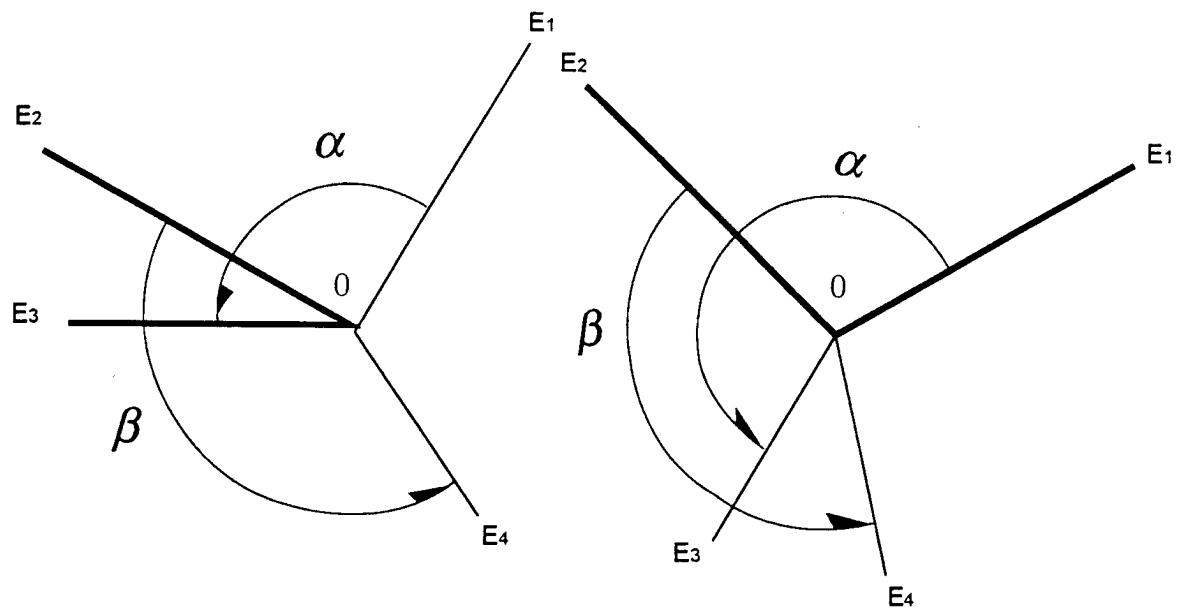
FIGURE 3.5b Entrance line finding
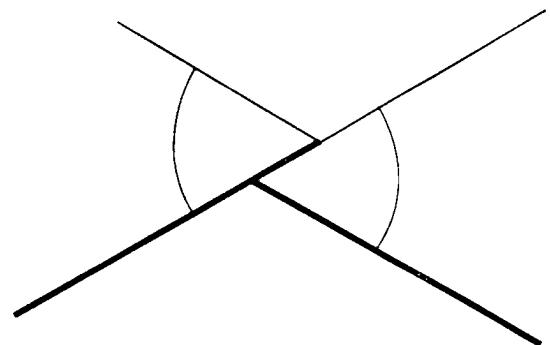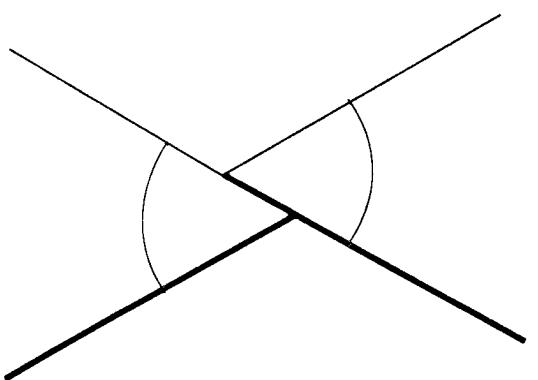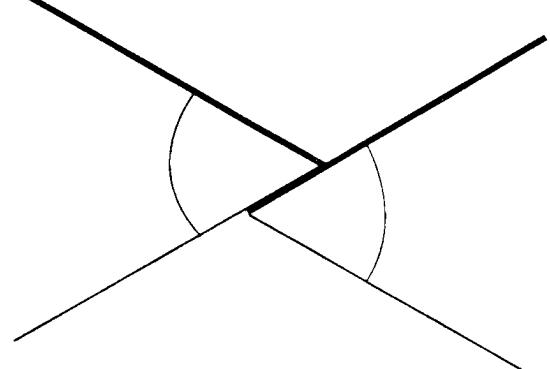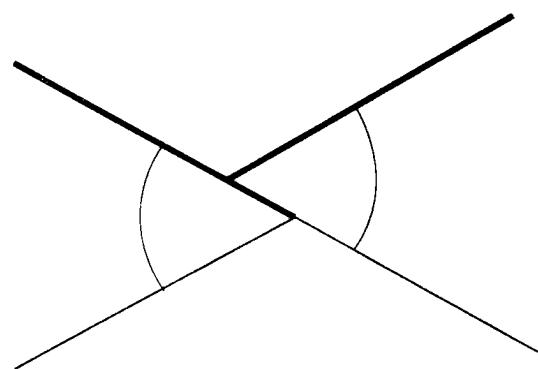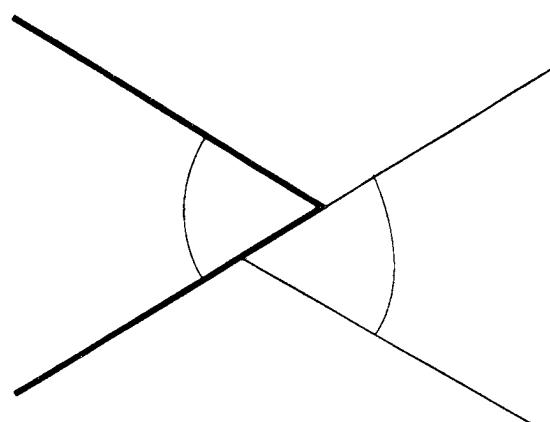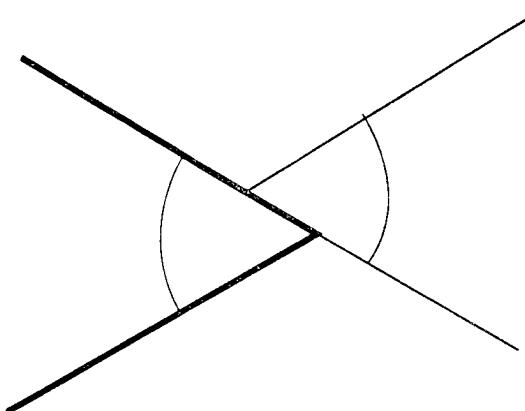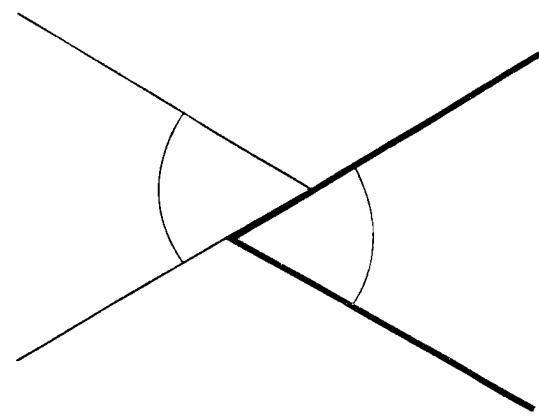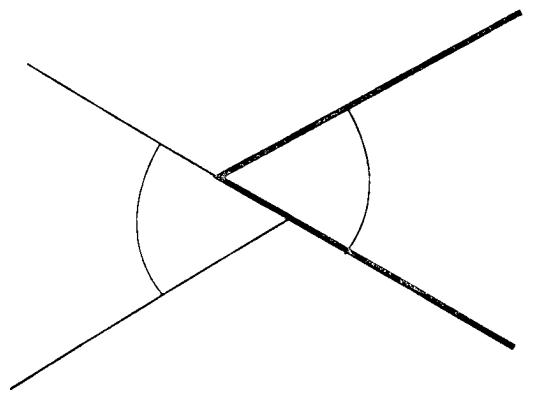
FIGURE 3.6 Entrance lines of parallel edges

[3] For the angle to edge contact, the only entrance line is the edge.

[4] For the edge to edge contact, the two edges are parallel, there is only one entrance line in this case, the entrance line can be any of the two edges.

As showing by Figure 3.6, the definition of entrance line is still correct even if edges of two sides of the contact are parallel or slightly penetrated.

The penetration judgment in different contacts are as following:

[1] For an angle to angle contact, if both angles are less than 180°, the two entrance line have to be passed simultaneously by the vertex of another angle.

[2] For an angle to angle contact, if a angle is larger than 180°, one of the two entrance lines has to be passed by the vertex of another angle.

[3] For an angle to edge contact, the only entrance line bas to be passed by the angle vertex.

For an angle to angle contact, penetration happens when there are no vertex entering the opposite angle as shown in Figure 3.7. Therefore the fact that if a vertex is in other angle is not a criteria of penetration. The penetration is related to the entrance lines.
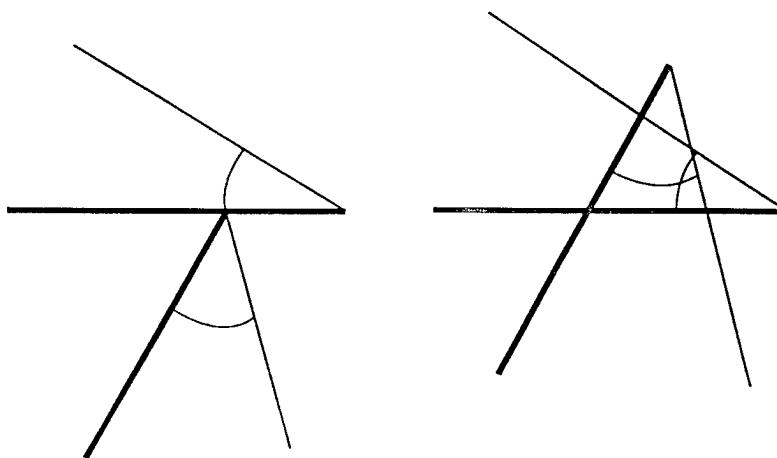
FIGURE 3.7 Using of entrance lines

In order to prevent the penetration of the two sides of the contacts, stiff springs are applied to some entrance lines:

[1] For angle to angle contact, if both angles are less than 180°, a stiff spring is attached between the vertex and its entrance line which has been passed first or has smaller passing distance d. For this type of contact, only one stiff spring is applied.

[2] For angle to angle contact, with a angle larger than 180°, if any one of the two entrance lines has been passed by the corresponding vertex of another angle, a stiff spring is applied along the normal to this edge. If two entrance lines have been passed, two stiff springs are applied to the two entrance edges.

[3] For angle to edge contact, one stiff spring is applied to the entrance edge.

The entrance distance d from P_1 to $P_2 P_3$ can be computed by the following formula,

$$
d = \frac{1}{l} \Delta = \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} < 0. \tag{3.3}
$$

$$
l = \sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2}
$$

If a contact is open in the beginning of a time step and is closed in the end of the same time step, the entrance time and position in this time step can be computed. Assume $t = 0$ in the beginning of the time step, and $t = 1$ in the end of the time step, and

$$
\Delta(t) = \begin{vmatrix} 1 & x_1 + t u_1 & y_1 + t v_1 \\ 1 & x_2 + t u_2 & y_2 + t v_2 \\ 1 & x_3 + t u_3 & y_3 + t v_3 \end{vmatrix}
$$

then

$$
\Delta(0) > 0 \quad \Delta(1) < 0
$$

The entrance time t_0 satisfies equation

$$
\Delta(t_0) = \begin{vmatrix} 1 & x_1 + t_0 u_1 & y_1 + t_0 v_1 \\ 1 & x_2 + t_0 u_2 & y_2 + t_0 v_2 \\ 1 & x_3 + t_0 u_3 & y_3 + t_0 v_3 \end{vmatrix} = 0 \tag{3.4}
$$

Neglecting the second order infinite small, t_0 can be computed by simpler formula:

$$t_0 = - \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \bigg/ \left(\begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix} + \begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix} \right) \tag{3.5}$$

The contact position (x_0, y_0) is

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_1 + u_1 t_0 \\ y_1 + v_1 t_0 \end{pmatrix} \tag{3.6}$$

These formulae are still correct even when the three points move simultaneously.

3.3 Contact Transfer to Next Time Step for General Covers

The computation of manifold method follows time steps. The closed contact points should go to next time step and find new representing contacts. The closed contacts of the previous time step will be transferred to the next time step, if the contacts are found in the same contact position. The entrance lines will be transferred in case it is possible. The angle to angle contacts and angle to edge contacts have different contact parameters and different stiff springs. The same contact may also has different contact parameters in different contact state.

The springs of closed angle to angle contacts:

[1] normal stiff spring,

The springs of closed angle to edge contacts in the sliding mode:

[1] normal stiff spring,

[2] pair of shear sliding forces,

The springs of the closed angle to edge contacts locked in both normal and shear directions

[1] normal stiff spring,

[2] shear stiff spring,

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The closed angle to angle contacts have the following contact, parameters:

[1] normal forces,

[2] normal displacements.

The closed angle to edge contacts in the sliding mode have the following contact parameters:

[1] normal forces,

[2] normal displacements,

[3] one of the two possible sliding directions,

[4] contact positions on the edges,

The closed angle to edge contacts locked in both normal and shear directions have the following contact parameters:

[1] normal forces,

[2] normal displacements,

[3] shear forces,

[4] shear displacements,

[5] contact positions on the edges.

Transferring the contacts to next step, an angle to angle contact may transferred to an angle to edge contact. Also an angle to edge contact may transferred to angle to angle contact. As transferring to different kind of contacts, some contact parameters may not be needed.

3.4 Normal Contact Matrices for General Covers

Assume P_1 is a vertex, P_2P_3 is the entrance line and (x_k, y_k) and (u_k, v_k) are the coordinates and displacement of P_k, $k = 1, 2, 3$ respectively. If points P_1, P_2 and P_3 rotate in the same sense as the rotation of ox to oy (see Figure 3.3), then the distance d from P_1 to

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line $P_2 P_3$ is

$$d = \frac{\Delta}{l} = \frac{1}{l} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix}, \tag{3.7}$$

$$l = \sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2}$$

d should be negative if P_1 passed edge $P_2\ P_3$

The step displacements

$$(u_i, v_i) \quad i = 1, 2, 3$$

is small as the results of small time step. The contact distance $\frac{\Delta}{l}$ is small from the definition of the contacts. Then

$$S_0 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad \begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix}, \quad \begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix}$$

are the first order infinite small, and

$$\begin{vmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2 \\ 1 & u_3 & v_3 \end{vmatrix}$$

is the second order infinite small.

$$\begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} + \begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix} + \begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix} + \begin{vmatrix} 1 & u_1 & v_1 \\ 1 & u_2 & v_2 \\ 1 & u_3 & v_3 \end{vmatrix} \tag{3.8}$$

Neglecting the second order infinite small, there is

$$\Delta \approx S_0 + \begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix} + \begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix}, \tag{3.9}$$

As

$$\begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix}$$

is first order infinite small,

$$d = \frac{1}{\sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2}} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix}$$

$$d \approx \frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} \tag{3.10}$$

Therefore the formula of l can be simplified as

$$l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \tag{3.11}$$

Considering only the first order small,

$$\Delta = S_0 + \left((y_2 - y_3) \quad (x_3 - x_2) \right) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$
$$+ \left((y_3 - y_1) \quad (x_1 - x_3) \right) \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \tag{3.12}$$
$$+ \left((y_1 - y_2) \quad (x_2 - x_1) \right) \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}.$$

And

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = [T(x_1, y_1)]\{D\},$$

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = [T(x_2, y_2)]\{D\},$$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = [T(x_3, y_3)]\{D\},$$

then

$$\Delta = S_0 + ((y_2 - y_3) \quad (x_3 - x_2)) [T(x_1, y_1)]\{D\},$$
$$+ ((y_3 - y_1) \quad (x_1 - x_3)) [T(x_2, y_2)]\{D\}, \tag{3.13}$$
$$+ ((y_1 - y_2) \quad (x_2 - x_1)) [T(x_3, y_3)]\{D\}.$$

Denote

$$\{H\} = \frac{1}{l}[T(x_1, y_1)]^T \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix}, \tag{3.14}$$

$$\{G\} = \frac{1}{l}[T(x_2, y_2)]^T \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix}$$
$$+ \frac{1}{l}[T(x_3, y_3)]^T \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix}, \tag{3.15}$$

and

$$\{H_i\} = \frac{1}{l}[T_i(x_1, y_1)]^T \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix},$$

$$\{G_i\} = \frac{1}{l}[T_i(x_2, y_2)]^T \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix}$$
$$+ \frac{1}{l}[T_i(x_3, y_3)]^T \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix},$$

so

$$\{H\} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_n \end{pmatrix} \qquad \{G\} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{pmatrix}$$

$$d = \{H\}^T\{D\} + \{G\}^T\{D\} + \frac{S_0}{l}. \tag{3.16}$$

The potential energy of the normal spring is

$$\Pi_s = \frac{p}{2}d^2$$

$$= \frac{p}{2}\left(\{H\}^T\{D\} + \{G\}^T\{D\} + \frac{S_0}{l}\right)^2$$

$$= \frac{p}{2}\left[\{D\}^T\{H\}\{H\}^T\{D\} + \{D\}^T\{G\}\{G\}^T\{D\} + 2\{D\}^T\{H\}\{G\}^T\{D\}\right.$$

$$\left. + 2\left(\frac{S_0}{l}\right)\{D\}^T\{H\} + 2\left(\frac{S_0}{l}\right)\{D\}^T\{G\} + \left(\frac{S_0}{l}\right)^2\right]. \tag{3.17}$$

Thus,

$$p\{H\}\{H\}^T$$

$$p\{H\}\{G\}^T$$

$$p\{G\}\{H\}^T$$

$$p\{G\}\{G\}^T$$

are the normal spring matrices of $[K]$

$$-p\left(\frac{S_0}{l}\right)\{H\}$$

$$-p\left(\frac{S_0}{l}\right)\{G\}$$

are the load matrices of $[F]$.

Then

$$
\begin{array}{llll}
p\{H_r\}\{H_s\}^T & \rightarrow & [K_{rs}], & r,s = 1,2,3,\ldots,n \\[2mm]
p\{H_r\}\{G_s\}^T & \rightarrow & [K_{rs}], & r,s = 1,2,3,\ldots,n \\[2mm]
p\{G_r\}\{H_s\}^T & \rightarrow & [K_{rs}], & r,s = 1,2,3,\ldots,n \\[2mm]
p\{G_r\}\{G_s\}^T & \rightarrow & [K_{rs}], & r,s = 1,2,3,\ldots,n \\[2mm]
-p\left(\dfrac{S_0}{l}\right)\{H_r\} & \rightarrow & \{F_r\}, & r = 1,2,3,\ldots,n \\[3mm]
-p\left(\dfrac{S_0}{l}\right)\{G_r\} & \rightarrow & \{F_r\}, & r = 1,2,3,\ldots,n
\end{array}
\tag{3.18}
$$

3.5 Shear Contact Matrix for General Covers

In Figure 3.8, point (x_0, y_0) is on the edge $P_2 P_3$. Point (x_0, y_0) is also the assumed contact point of vertex P_1. The shear spring on the direction $P_2 P_3$ connects vertices P_1 and P_0.

Let
$$l = \sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2},$$

the shear displacement of P_0 and P_1 along line $P_2 P_3$ is

$$d = \frac{1}{l} \overrightarrow{P_0 P_1} \cdot \overrightarrow{P_2 P_3} \tag{3.19}$$

$$= \frac{1}{l} \left((x_1 + u_1) - (x_0 + u_0) \quad (y_1 + v_1) - (y_0 + v_0) \right) \begin{pmatrix} (x_3 + u_3) - (x_2 + u_2) \\ (y_3 + v_3) - (y_2 + v_2) \end{pmatrix}.$$

The step displacements

$$(u_i, v_i) \quad i = 1, 2, 3$$

is small as the results of small time step. The contact distance $\frac{\Delta}{l}$ is small from the definition of the contacts. Then the projection of the displacements (u_0, v_0) and (u_1, v_1) of points P_0 and P_1 on vector $\overrightarrow{P_2 P_3}$

$$\left(x_3 + u_3 - x_2 - u_2 \quad y_3 + v_3 - y_2 - v_2 \right) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

$$\left(x_3 + u_3 - x_2 - u_2 \quad y_3 + v_3 - y_2 - v_2 \right) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

are small. And

$$\left(u_3 - u_2 \quad v_3 - v_2 \right) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

is second order small.

The contact distance is small from the definition of the contacts. Then $P_0 P_1$ is small, and

$$S_0 = \left(x_1 - x_0 \quad y_1 - y_0 \right) \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}$$

is small.

$$\begin{pmatrix} x_1 - x_0 & y_1 - y_0 \end{pmatrix} \begin{pmatrix} u_3 - u_2 \\ v_3 - v_2 \end{pmatrix}$$

is second order small.

$$d =$$

$$\frac{1}{\sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2}}$$

$$\begin{pmatrix} (x_1 + u_1) - (x_0 + u_0) & (y_1 + v_1) - (y_0 + v_0) \end{pmatrix} \begin{pmatrix} (x_3 + u_3) - (x_2 + u_2) \\ (y_3 + v_3) - (y_2 + v_2) \end{pmatrix}$$

$$d \approx$$

$$\frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}}$$

$$\begin{pmatrix} (x_1 + u_1) - (x_0 + u_0) & (y_1 + v_1) - (y_0 + v_0) \end{pmatrix} \begin{pmatrix} (x_3 + u_3) - (x_2 + u_2) \\ (y_3 + v_3) - (y_2 + v_2) \end{pmatrix}$$

Therefore the formula of l can be simplified as

$$l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} \tag{3.20}$$

$$d \approx \frac{S_0}{l} + \frac{1}{l} \begin{pmatrix} x_3 - x_2 & y_3 - y_2 \end{pmatrix} \begin{pmatrix} u_1 - u_0 \\ v_1 - v_0 \end{pmatrix},$$

$$d \approx \frac{S_0}{l} + \frac{1}{l} \begin{pmatrix} x_3 - x_2 & y_3 - y_2 \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

$$+ \frac{1}{l} \begin{pmatrix} x_2 - x_3 & y_2 - y_3 \end{pmatrix} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{3.21}$$

Denote

$$\{H\} = \frac{1}{l} [T(x_1, y_1)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}, \tag{3.22}$$

$$\{G\} = \frac{1}{l} [T(x_0, y_0)]^T \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix}, \tag{3.23}$$

and

$$\{H_i\} = \frac{1}{l}[T_i(x_1, y_1)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},$$

$$\{G_i\} = \frac{1}{l}[T_i(x_0, y_0)]^T \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix},$$

so

$$\{H\} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_n \end{pmatrix} \qquad \{G\} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{pmatrix}$$

Then (3.21) becomes

$$d = \{H\}^T \{D\} + \{G\}^T \{D\} + \frac{S_0}{l}. \tag{3.24}$$

The potential energy of the shear spring is

$$\begin{aligned}
\Pi_s &= \frac{p}{2} d^2 \\
&= \frac{p}{2} \left(\{H\}^T \{D\} + \{G\}^T \{D\} + \frac{S_0}{l} \right)^2 \\
&= \frac{p}{2} \Big[\{D\}^T \{H\} \{H\}^T \{D\} + \{D\}^T \{G\} \{G\}^T \{D\} + 2 \{D\}^T \{H\} \{G\}^T \{D\} \\
&\quad + 2 \left(\frac{S_0}{l} \right) \{D\}^T \{H\} + 2 \left(\frac{S_0}{l} \right) \{D\}^T \{G\} + \left(\frac{S_0}{l} \right)^2 \Big]. \tag{3.25}
\end{aligned}$$

Thus,

$$p\{H\}\{H\}^T$$
$$p\{H\}\{G\}^T$$
$$p\{G\}\{H\}^T$$
$$p\{G\}\{G\}^T$$

are the shear spring matrices of $[K]$

$$-p\left(\frac{S_0}{l}\right)\{H\}$$

$$-p\left(\frac{S_0}{l}\right)\{G\}$$

are the load matrices of $[F]$ from shear spring.

$$
\begin{aligned}
p\{H_r\}\{H_s\}^T &\rightarrow &[K_{rs}], &\quad r,s = 1,2,3,\ldots,n \\
p\{H_r\}\{G_s\}^T &\rightarrow &[K_{rs}], &\quad r,s = 1,2,3,\ldots,n \\
p\{G_r\}\{H_s\}^T &\rightarrow &[K_{rs}], &\quad r,s = 1,2,3,\ldots,n \\
p\{G_r\}\{G_s\}^T &\rightarrow &[K_{rs}], &\quad r,s = 1,2,3,\ldots,n \\
-p\left(\frac{S_0}{l}\right)\{H_r\} &\rightarrow &\{F_r\}, &\quad r = 1,2,3,\ldots,n \\
-p\left(\frac{S_0}{l}\right)\{G_r\} &\rightarrow &\{F_r\}, &\quad r = 1,2,3,\ldots,n
\end{aligned}
\tag{3.26}
$$

3.6 Friction Force Matrix for General Covers

Friction forces are treated as loading forces in forward analysis, therefore the coefficient matrix of the equations will still be symmetric.

When Coulomb's law allows sliding between two sides of boundary contacts, there exist friction forces in two sliding sides if the friction angle ϕ is not zero.

As shown in Figure 3.8, P_1 is on one side of a contact and P_2, P_3, P_0 are on the other side of the same contact. The friction force is calculated from the normal contact compressive force and the direction of the friction force is depending on the movement of P_1 relative to P_0 in the direction from P_2 to P_3. Let p be the stiffness of normal contact spring, then the friction force

$$\mathcal{F} = p \cdot d \cdot s \cdot \tan(\phi), \tag{3.27}$$

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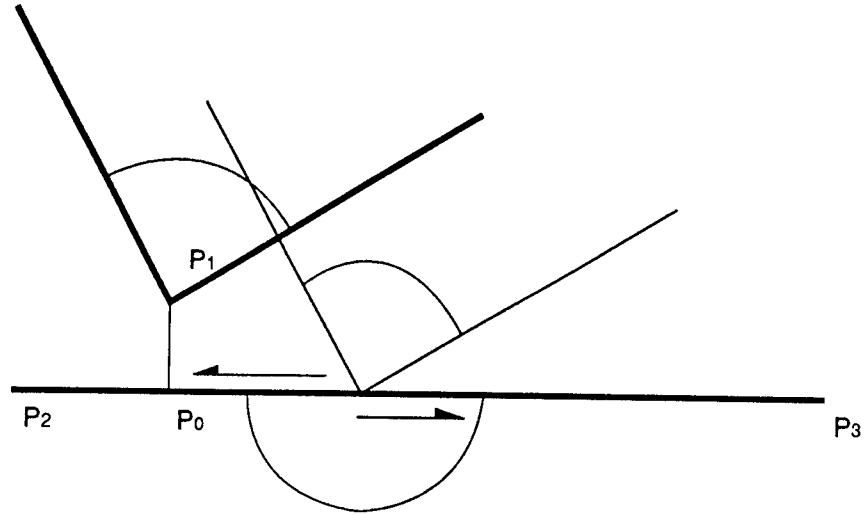
FIGURE 3.8 Position of shear spring

where

d is the normal penetration distance,

$\tan(\phi)$ is the friction coefficient,

$s = sgn$ (movement of P_1 relative to P_0 in the direction from P_2 to P_3) , and

$$sgn(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

The friction force $\mathcal{F}$ is along the direction

$$\frac{1}{l}\Big((x_3 - x_2) \quad (y_3 - y_2) \Big),$$

and

$$l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$$

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is the length of line $\overline{P_2 P_3}$.

Then the potential energy of friction force $\mathcal{F}$ at P_1 side is

$$
\begin{aligned}
\Pi_f &= \frac{\mathcal{F}}{l} \begin{pmatrix} u_1 & v_1 \end{pmatrix} \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix} \\
&= \mathcal{F}\{D\}^T [T(x_1, y_1)]^T \begin{pmatrix} (x_3 - x_2)/l \\ (y_3 - y_2)/l \end{pmatrix} \\
&= \mathcal{F}\{D\}^T \{H\},
\end{aligned}
\tag{3.28}
$$

Denote

$$
\{H\} = \frac{1}{l} [T(x_1, y_1)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},
\tag{3.29}
$$

and

$$
\{H_i\} = \frac{1}{l} [T_i(x_1, y_1)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},
$$

so

$$
\{H\} = \begin{pmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_n \end{pmatrix}
$$

Then

$$
-\mathcal{F}\{H\} = -\mathcal{F} \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \\ \vdots \\ \{H_n\} \end{pmatrix}
\tag{3.30}
$$

is the loading matrix for point P_1; and

$$-\mathcal{F}\{H_r\} \qquad \rightarrow \qquad \{F_r\}, \qquad r = 1, 2, 3, \ldots, n$$

Similarly, the potential energy of friction force $\mathcal{F}$ at P_0 side is

$$
\begin{aligned}
\Pi_f &= -\frac{\mathcal{F}}{l} \begin{pmatrix} u_0 & v_0 \end{pmatrix} \begin{pmatrix} (x_3 - x_2) \\ (y_3 - y_2) \end{pmatrix} \\
&= -\mathcal{F}\{D_j\}^T [T_j(x_0, y_0)]^T \begin{pmatrix} (x_3 - x_2)/l \\ (y_3 - y_2)/l \end{pmatrix} \\
&= -\mathcal{F}\{D_j\}^T \{G\},
\end{aligned}
\tag{3.31}
$$

Denote

$$\{G\} = \frac{1}{l} [T(x_0, y_0)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}, \tag{3.32}$$

and

$$\{G_i\} = \frac{1}{l} [T_i(x_0, y_0)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},$$

so

$$\{G\} = \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{pmatrix}$$

Then

$$+\mathcal{F}\{G\} = +\mathcal{F} \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \\ \vdots \\ \{G_n\} \end{pmatrix} \tag{3.33}$$

is the loading matrix for point P_0; and

$$+\mathcal{F}\{G_r\} \qquad \rightarrow \qquad \{F_r\}, \qquad r = 1, 2, 3, \ldots, n$$

Chapter 4

Finite Element Covers of Manifold Method

4.1 Finite Covers Formed by Finite Element Nodes and Physical Boundaries

The manifold method can perform the computations of finite element method for continuous material. After transferring the finite element mesh to finite covers of manifold method, the joint lines can be input as new physical mesh, then the same finite element mesh can compute joints in the same material volume.

The finite element meshes can be used to define finite covers for manifold method. Considering any node, all elements having this node form a mathematical cover (called "star" in algebraic topology).

In Figure 4.1 and 4.2, the mathematical cover V_5 of node 5 has three elements 2 4 5, 2 5 3 and 3 5 6. The mathematical cover V_1 of node 1 has only one element 1 2 3 which is the only element having node 1.

mathematical covers of Figure 4.1 and Figure 4.2

node	element	element	element	element	element	element
1	1,2,3					
2	1,2,3	2,4,5	2,5,3			
3	1,2,3	2,5,3	3,5,6			
4	2,4,5					
5	2,4,5	2,5,3	3,5,6			
6	3,5,6					

Figure 4.3 shows the mathematical covers of nodes $1, 2, 4, 5$, mathematical covers of all nodes are listed in the table.
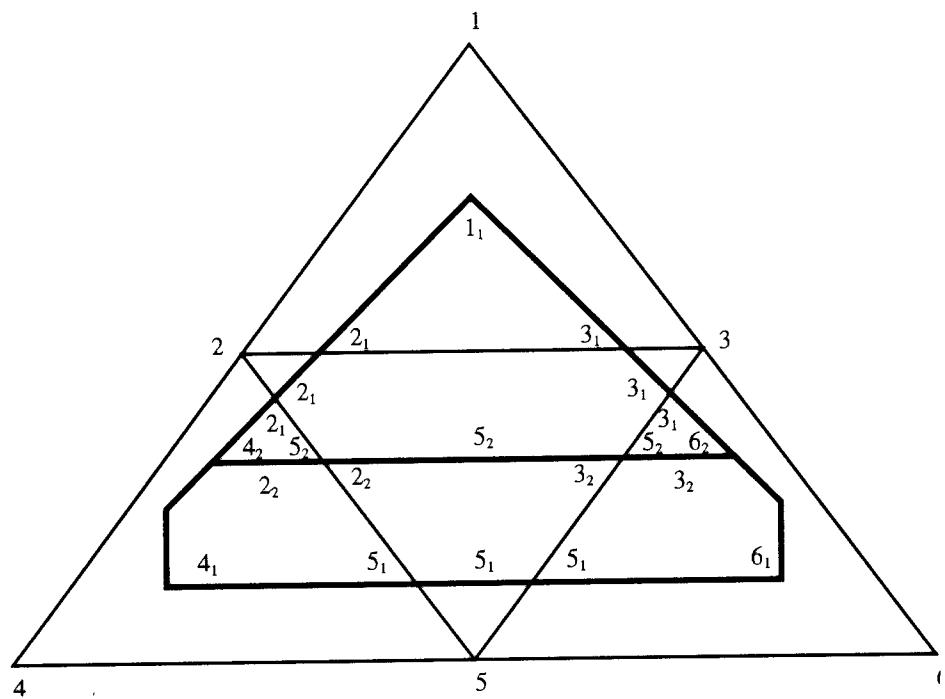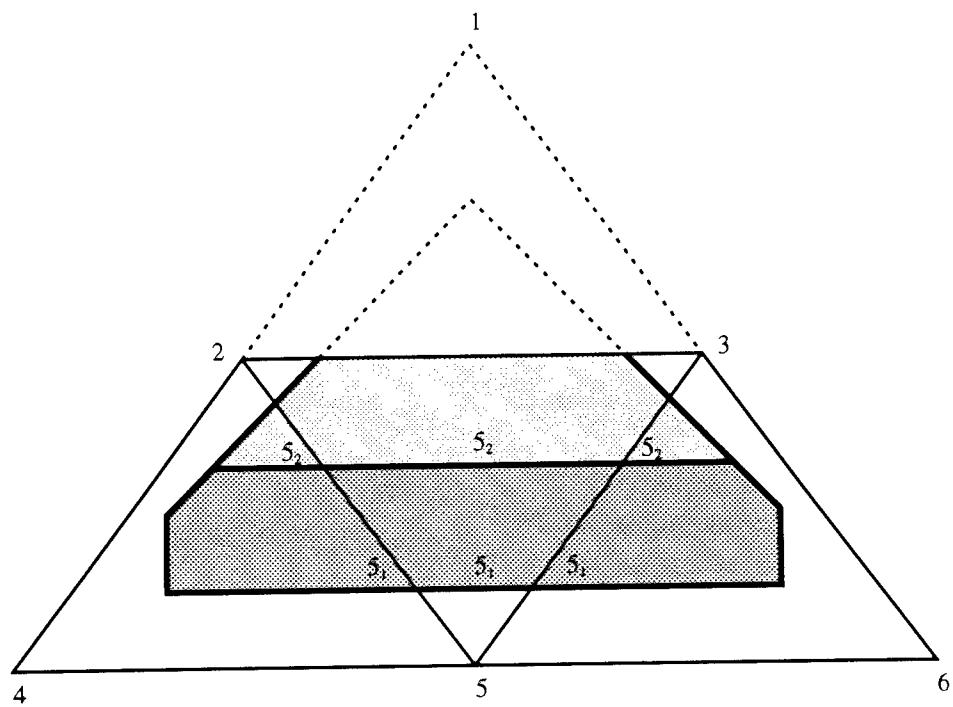
FIGURE 4.1 FEM covers on two blocks
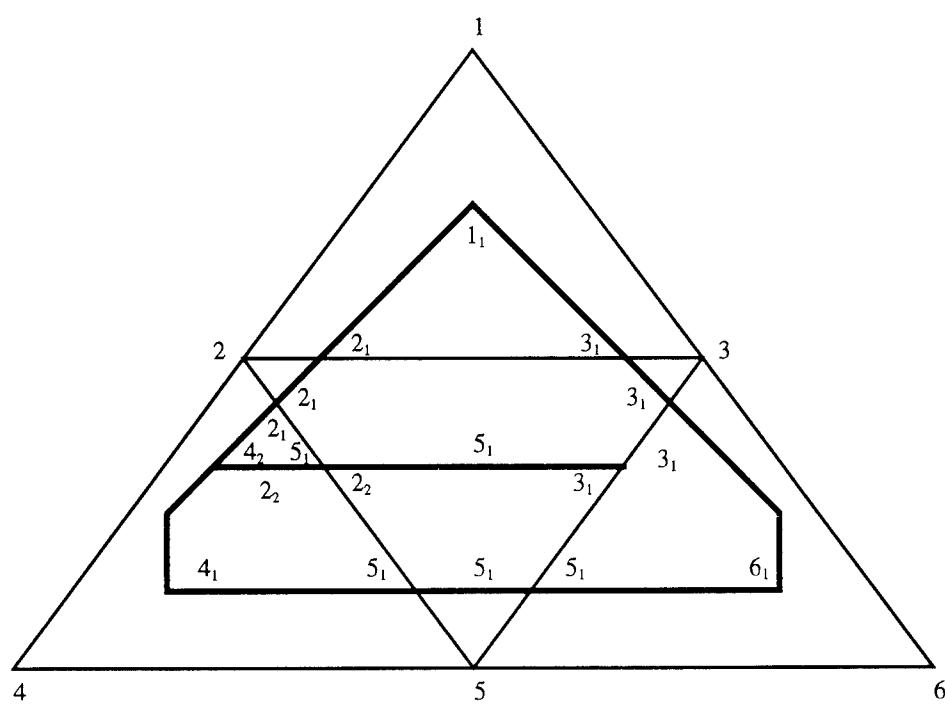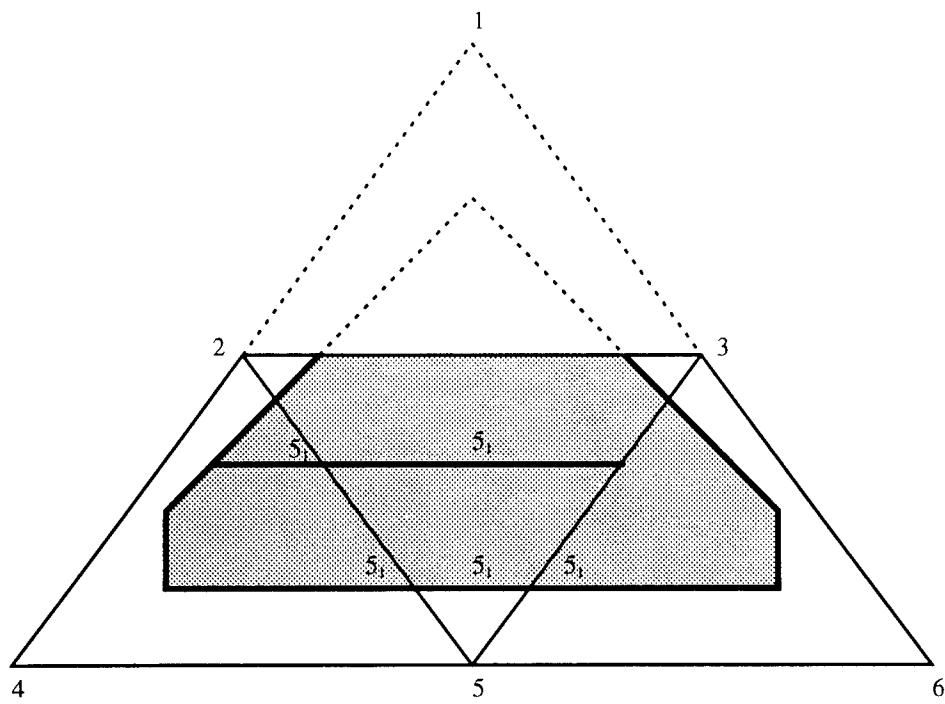
FIGURE 4.2 FEM covers with a joint
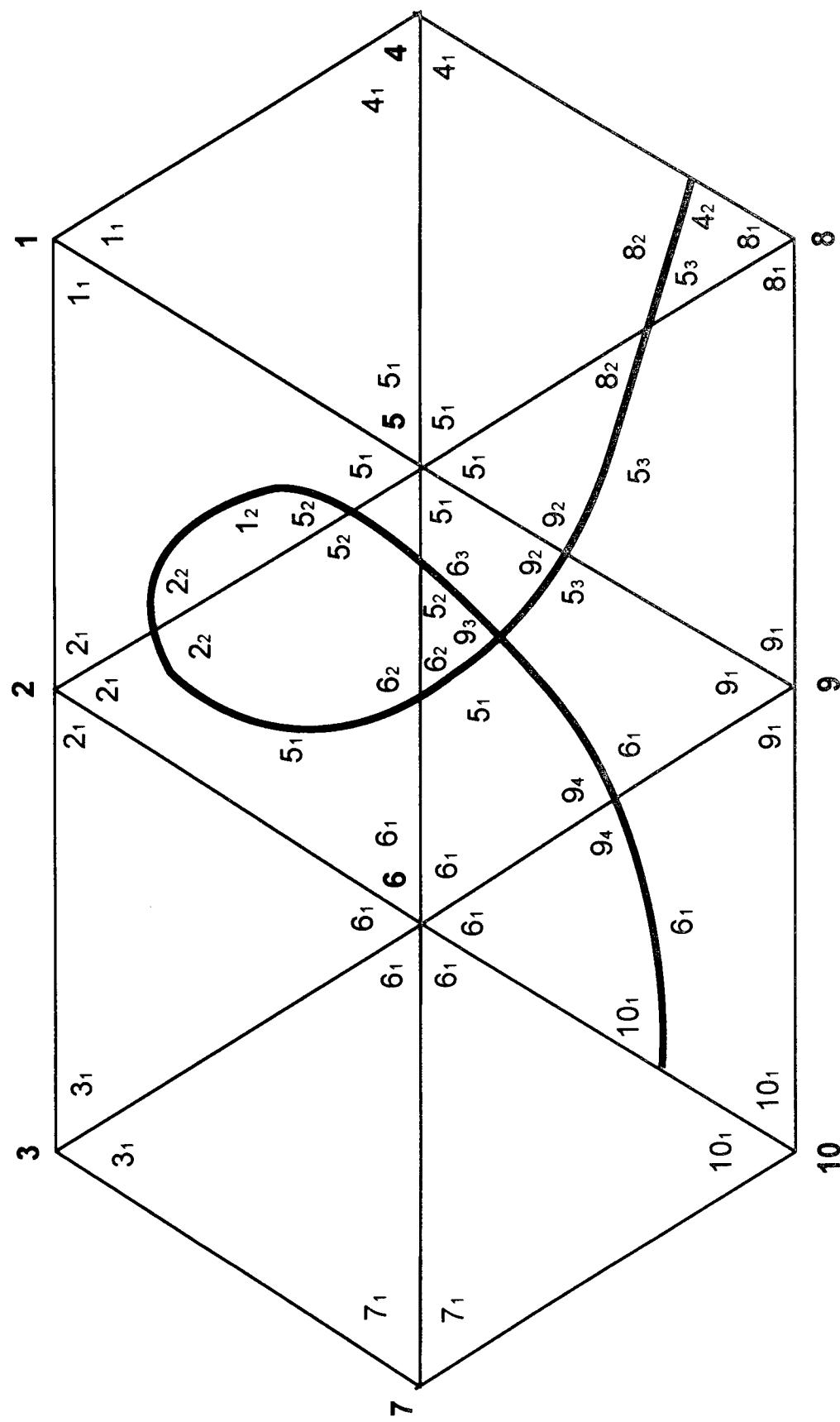
FIGURE 4.3 FEM covers with a complex joint

mathematical covers of Figure 4.3

node	element	element	element	element	element	element
1	1,2,5	1,5,4				
2	2,3,6	2,6,5	2,5,1			
3	2,3,6	3,7,6				
4	1,5,4	4,5,8				
5	1,5,4	1,2,3	2,6,5	5,6,9	5,9,8	4,5,8
6	2,6,5	2,3,6	3,7,6	6,7,10	6,10,9	6,9,5
7	3,7,6	6,7,10				
8	4,5,8	5,9,8				
9	5,9,8	5,6,9	6,10,9			
10	6,7,10	6,10,9				

Any original finite element is the common area of the mathematical cover of its nodes. In Figure 4.1 and Figure 4.2, mathematical cover V_5 of node 5 is the area defined by the polygon 2 4 5 6 3; mathematical cover V_2 of node 2 is the area defined by the polygon 1 2 4 5 3; mathematical cover V_3 of node 3 is the area defined by the polygon 1 2 5 6 3. The common part of mathematical covers V_5, V_2 and V_3 are original finite element 5 2 3.

The physical mesh of Figure 4.1 and 4.2 are the thick lines, which are the boundaries and the fractures of the material volume. The physical covers or covers are defined as following:

[1] The region of physical cover is the materials contained in the mathematical cover, or mathematically speaking the intersection of the mathematical cover and the material field.

[2] If the material boundaries, block boundaries or fractures divided the mathematical cover to totally isolated regions, each region is a physical cover. Therefore the physical covers are the subdivision of the mathematical covers.

The mathematical covers and divided physical covers of Figure 4.1, 4.2 and 4.3 are listed by the tables.

the mathematical covers and physical covers of Figure 4.1

math cover	physical cover	physical cover	physical cover	physical cover
V_1	1_1			

V_2	2_1	2_2
V_3	3_1	3_2
V_4	4_1	4_2
V_5	5_1	5_2
V_6	6_1	6_2

the mathematical covers and physical covers of Figure 4.2

math cover	physical cover	physical cover	physical cover	physical cover
V_1	1_1			
V_2	2_1	2_2		
V_3	3_1			
V_4	4_1	4_2		
V_5	5_1			
V_6	6_1			

the mathematical covers and physical covers of Figure 4.3

math cover	physical cover	physical cover	physical cover	physical cover
V_1	1_1	1_2		
V_2	2_1	2_2		
V_3	3_1			
V_4	4_1	4_2		
V_5	5_1	5_2	5_3	
V_6	6_1	6_2	6_3	
V_6	7_1			
V_6	8_1	8_2		
V_6	9_1	9_2	9_3	
V_6	10_1			

4.2 Elements as The Common Part of Node Covers

The elements of the manifold are the common regions or the intersections of the physical covers. Each point inside the material boundary lies in a "element" which is a common part of exactly three physical covers in the triangle finite element case. The relations of the original finite element terminology and its manifold generalization are

dimensions	finite element method	manifold method
0-d to 2-d	node	physical cover

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1-d to 1-d	edge	two cover intersection
2-d to 0-d	element	intersection of covers

Under manifold method, the "elements" and "nodes" here are the extensions of their FEM counterparts. Using the new nodes and elements, the joints can open and slide, the blocks can move away and the continuous area of the material body can still be connected.

The proof of these important conclusions come directly from the definition of the finite cover systems and the cover functions and global displacement functions of the manifold method.

A original finite element can be divided to several elements of manifold method by the block boundaries, joints or fractures. Before the deformation, the nodes share the same position. It can be understood as many layers divided by discontinuities on the original simple finite element mesh.

The manifold elements of Figure 4.1, Figure 4.2 and Figure 4.3 are listed bellow.

Physical covers or nodes of elements of Figure 4.1

element number	physical cover	physical cover	physical cover
1,2,3	$1_1, 2_1, 3_1$		
2,4,5	$2_2, 4_1, 5_1$	$2_1, 4_2, 5_2$	
2,5,3	$2_1, 5_2, 3_1$	$2_2, 5_1, 3_2$	
3,5,6	$3_1, 5_2, 6_2$	$3_2, 5_1, 6_1$	

Physical covers or nodes of elements of Figure 4.2

element number	physical cover	physical cover	physical cover
1,2,3	$1_1, 2_1, 3_1$		
2,4,5	$2_2, 4_1, 5_1$	$2_1, 4_2, 5_1$	
2,5,3	$2_1, 5_1, 3_1$	$2_2, 5_1, 3_1$	
3,5,6	$3_1, 5_1, 6_1$		

Physical covers or nodes of elements of Figure 4.3

original element	physical cover	physical cover	physical cover
1,5,4	$1_1, 5_1, 4_1$		
1,2,5	$1_1, 2_1, 5_1$	$1_2, 2_2, 5_2$	
2,6,5	$2_1, 6_1, 5_1$	$2_2, 6_2, 5_2$	$2_1, 6_3, 5_1$
2,3,6	$2_1, 3_1, 6_1$		

3,7,6	$3_1,7_1,6_1$		
4,5,8	$4_1,5_1,8_2$	$4_2,5_3,8_1$	
5,9,8	$5_1,9_2,8_2$	$5_3,9_1,8_1$	
5,6,9	$5_3,6_1,9_1$	$5_1,6_3,9_2$	$5_2,6_2,9_3$
5,6,9	$5_1,6_1,9_4$		
6,10,9	$6_1,10_1,9_4$	$6_1,10_1,9_1$	
6,7,10	$6_1,7_1,10_1$		

In Figure 4.1, the only joint inside the material divided completely the material to two disconnected parts, any manifold element divided by this joint have completely different nodes or physical cover numbers. Therefore the two manifold elements are free to move independently.

element separation along joint in Figure 4.1

original element	manifold element	manifold element
2,4,5	$2_2,4_1,5_1$	$2_1,4_2,5_2$
2,5,3	$2_1,5_2,3_1$	$2_2,5_1,3_2$
3,5,6	$3_1,5_2,6_2$	$3_2,5_1,6_1$

In Figure 4.2, the only joint inside the material divided partly the material, any manifold element divided by this joint have partly different nodes or physical cover numbers. Therefore the two manifold elements can have different movements.

element separation along joint in Figure 4.2

original element	manifold element	manifold element
2,4,5	$2_2,4_1,5_1$	$2_1,4_2,5_1$
2,5,3	$2_1,5_1,3_1$	$2_2,5_1,3_1$
3,5,6	$3_1,5_1,6_1$	

If two manifold elements share a edge of original finite element mesh, as this edge is not a joint, the two manifold elements have the same nodes on the edge:

element connection along edges in Figure 4.1

nodes of edge	manifold element	manifold element
2,3	$1_1,2_1,3_1$	$2_1,5_2,3_1$
2,5	$2_1,4_2,5_2$	$2_1,5_2,3_1$
2,5	$2_2,4_1,5_1$	$2_2,5_1,3_2$

| 3,5 | $2_1,5_2,3_1$ | $3_1,5_2,6_2$ |
| 3,5 | $2_2,5_1,3_2$ | $3_2,5_1,6_1$ |

element connection along edges in Figure 4.2

nodes of edge	manifold element	manifold element
2,3	$1_1,2_1,3_1$	$2_1,5_1,3_1$
2,5	$2_1,4_2,5_1$	$2_1,5_1,3_1$
2,5	$2_2,4_1,5_1$	$2_2,5_1,3_1$
3,5	$2_1,5_1,3_1$	$3_1,5_1,6_1$
3,5	$2_2,5_1,3_1$	$3_1,5_1,6_1$

Using the manifold definition of nodes and elements, the following important conclusions can be proved and also can be seen directly from Figure 4.1, 4.2 and 4.3:

[1] the elements are irregularly shaped;

[2] each element has three physical cover numbers;

[3] these three covers have one element as their common area;

[4] the three covers can be seen as three "nodes" of the element;

[5] the adjacent elements have the same nodes along the common edge;

[6] two elements divided by fractures or boundaries have different nodes.

4.3 Cover Functions and Weight Functions of Finite Element Mesh

The displacement functions are independent from the material boundary. If the material occupies only part of the element, the displacement functions are still the same. For a triangular element, there are three covers containing this element corresponding three nodes. Therefore each element is the common region of three covers of its three nodes.

For a element, the weight functions are computed, denote i_i : (x_i, y_i) the coordinates of nodes $i = 1, 2, 3$, and the related nodal displacements as follows:

$$
\begin{array}{ccc}
coordinates & \rightarrow & displacements \\
i_1 : (x_1, y_1) & \rightarrow & (u_1, v_1), \\
i_2 : (x_2, y_2) & \rightarrow & (u_2, v_2), \\
i_3 : (x_3, y_3) & \rightarrow & (u_3, v_3).
\end{array}
$$

The displacement field can be described as:

$$
\begin{aligned}
u &= a_1 + b_1 x + c_1 y, \\
v &= a_2 + b_2 x + c_2 y,
\end{aligned} \tag{4.1}
$$

The nodal displacements are

$$
\begin{aligned}
u_1 &= a_1 + b_1 x_1 + c_1 y_1, \\
u_2 &= a_1 + b_1 x_2 + c_1 y_2, \\
u_3 &= a_1 + b_1 x_3 + c_1 y_3, \\
v_1 &= a_2 + b_2 x_1 + c_2 y_1, \\
v_2 &= a_2 + b_2 x_2 + c_2 y_2, \\
v_3 &= a_2 + b_2 x_3 + c_2 y_3,
\end{aligned} \tag{4.2}
$$

and by matrix notation equation (4.2) becomes

$$
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \tag{4.3}
$$

and

$$
\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}. \tag{4.4}
$$

Then

$$
\begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},
$$

$$\begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

and

$$u = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}, \qquad v = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix},$$

thus,

$$u = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \tag{4.5}$$

and

$$v = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}. \tag{4.6}$$

Denote

$$\begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} = \begin{pmatrix} 1 & x & y \end{pmatrix} \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1},$$

and

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} = \begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1},$$

and the determinant

$$\Delta = \begin{vmatrix} 1 & x_2 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

is two times the area of the element.

$$\begin{pmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{pmatrix}^{-1} = \frac{1}{\Delta} \begin{pmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} & - \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} & + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ - \begin{vmatrix} 1 & y_2 \\ 1 & y_3 \end{vmatrix} & + \begin{vmatrix} 1 & y_1 \\ 1 & y_3 \end{vmatrix} & - \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} \\ + \begin{vmatrix} 1 & x_2 \\ 1 & x_3 \end{vmatrix} & - \begin{vmatrix} 1 & x_1 \\ 1 & x_3 \end{vmatrix} & + \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} \end{pmatrix},$$

then

$$\begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{pmatrix}. \tag{4.7}$$

$$(\, f_1 \quad f_2 \quad f_3 \,) = (\, 1 \quad x \quad y \,) \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix},$$

and

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_{11} + f_{21}x + f_{31}y \\ f_{12} + f_{22}x + f_{32}y \\ f_{13} + f_{23}x + f_{33}y \end{pmatrix}. \tag{4.8}$$

$$u = (\, f_1 \quad f_2 \quad f_3 \,) \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \qquad v = (\, f_1 \quad f_2 \quad f_3 \,) \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{4.9}$$

In the finite element, the cover functions are the constant function u_i, v_i over the whole cover. The cover functions $(u_i(x,y), v_i(x,y))$ defined on physical cover U_i

$$\left\{ \begin{array}{c} u_i(x,y) \\ v_i(x,y) \end{array} \right\} = \begin{pmatrix} u_i \\ v_i \end{pmatrix} \quad (x,y) \in U_i \tag{4.10}$$

The weight function $w_i(x,y)$ is the shape function of finite element method.

$$w_i(x,y) = f_i(x,y) \quad i = 1, 2, \ldots, n$$

The meaning of the weight functions $w_i(x,y)$ is weighted average, which is to take a percentage from each node displacements u_i, v_i. For each element, the summation of three weight function of three nodes are 1.

$$\sum_{(x,y) \in U_j} w_j(x,y) = 1. \tag{4.11}$$

Equation (4.11) can be derived from equation (4.7),

$$f_{11} + f_{12} + f13 = 1$$
$$f_{21} + f_{22} + f23 = 0$$
$$f_{31} + f_{32} + f33 = 0$$

$$w_1(x,y) + w_2(x,y) + w_3(x,y)$$
$$= f_1(x,y) + f_2(x,y) + f_3(x,y) = 1$$

4.4 Finite Element Global Functions for Continuous and Discontinuous Materials

Using the weight functions $w_i(x,y)$ a global function on the whole physical cover system is defined from the cover functions

$$\left\{ \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right\} = \left(\begin{array}{c} \sum_{i=1}^{n} w_i(x,y)u_i \\ \sum_{i=1}^{n} w_i(x,y)v_i \end{array} \right) \tag{4.12}$$

$$\left\{ \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right\} = \left(\begin{array}{c} \sum_{i=1}^{n} f_i(x,y)u_i \\ \sum_{i=1}^{n} f_i(x,y)v_i \end{array} \right) \tag{4.13}$$

Then

$$\left\{ \begin{array}{c} u(x,y) \\ v(x,y) \end{array} \right\} = \left(\begin{array}{cccccc} f_1 & 0 & f_2 & 0 & f_3 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 \end{array} \right) \left(\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{array} \right)$$

$$= [T_e]\{D_e\}, \tag{4.14}$$

where

$$[T_e] = (\, [T_1] \quad [T_2] \quad [T_3] \,), \qquad \{D_e\} = \left(\begin{array}{c} \{D_1\} \\ \{D_2\} \\ \{D_3\} \end{array} \right),$$

and

$$[T_i] = \left(\begin{array}{cc} f_i(x,y) & 0 \\ 0 & f_i(x,y) \end{array} \right), \qquad \{D_i\} = \left(\begin{array}{c} u_i \\ v_i \end{array} \right), \qquad i = 1, 2, 3.$$

$$\left(\begin{array}{c} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{array} \right) = \left(\begin{array}{c} f_{11} + f_{21}x + f_{31}y \\ f_{12} + f_{22}x + f_{32}y \\ f_{13} + f_{23}x + f_{33}y \end{array} \right).$$

The cover functions of the triangular finite element method are constants, the global functions are linear functions of the coordinates (x, y).

For manifold method, the cover function can be normal two dimensional series, where the unknowns are the coefficients of the series.

$$\begin{Bmatrix} u_i(x,y) \\ v_i(x,y) \end{Bmatrix} = \begin{pmatrix} \sum_{j=1}^{m} u_{ij} s_j(x,y) \\ \sum_{j=1}^{m} v_{ij} s_j(x,y) \end{pmatrix} \qquad (4.15)$$

The global displacement function is the combination of locally defined series.

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix}$$
$$= \begin{pmatrix} \sum_{i=1}^{n} w_i(x,y) u_i(x,y) \\ \sum_{i=1}^{n} w_i(x,y) v_i(x,y) \end{pmatrix}$$
$$= \begin{pmatrix} \sum_{i=1}^{n} \sum_{j=1}^{m} u_{ij} s_j(x,y) w_i(x,y) \\ \sum_{i=1}^{n} \sum_{j=1}^{m} v_{ij} s_j(x,y) w_i(x,y) \end{pmatrix} \qquad (4.16)$$

The submatrix form of the global displacement function is

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [T_e]\{D_e\}, \qquad (4.17)$$

where

$$[T_e] = \left([T_1] \quad [T_2] \quad [T_3] \right), \qquad \{D_e\} = \begin{pmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \end{pmatrix},$$

and

$$[T_i] =$$

$$\begin{pmatrix} f_{i1} & 0 & f_{i2} & 0 & f_{i3} & 0 & \cdots & \cdots & f_{nm} & 0 \\ 0 & f_{i1} & 0 & f_{i2} & 0 & f_{i3} & \cdots & \cdots & 0 & f_{nm} \end{pmatrix} \begin{pmatrix} u_{i1} \\ v_{i1} \\ u_{i2} \\ v_{i2} \\ u_{i3} \\ v_{i3} \\ \vdots \\ \vdots \\ u_{im} \\ v_{im} \end{pmatrix}$$

$$i = 1, 2, 3.$$

$$f_{ij}(x,y) = s_j(x,y)w_i(x,y) \qquad i = 1,2,3, \qquad j = 1,2,3,\ldots,m.$$

4.5 Finite Element Equilibrium Equations Based on Manifold Method

The total potential energy Π is the summation over all the potential energy sources: individual stresses and forces. In the following, the potential energy of each force or stress and their differentiations are computed separately.

[1] the strain potential energy Π_e produces the stiffness matrix,

[2] the potential energy Π_σ of initial stresses produces the initial stress matrix,

[3] the potential energy Π_p of point load produces the point load matrix,

[4] the potential energy Π_w of body load produces the body load matrix,

[5] the potential energy Π_i of inertia produces mass matrix,

[6] the strain potential energy Π_s of contact springs produces contact matrices,

[7] the potential energy Π_f of friction forces.

There are n physical covers or n nodes. For the two dimensional triangle elements j, there are three physical covers or nodes per element j_1, j_2, j_3. Each physical cover or node i has two unknowns (u_i, v_i)

$$[D_i] = \begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} d_{i1} \\ d_{i2} \end{pmatrix}$$

Therefore the unknown $[D_i]$ of cover i is a 2×1 submatrix. The coefficient matrix should formed by the 2×2 submatrices $[K_{ij}]$ in the equation (4.18). The force term F_i on cover i is also a 2×1 submatrix due to the dimension of D_i

Adding all potential energy together, the total potential energy has the form

$$\Pi = \frac{1}{2} \begin{pmatrix} D_1^T & D_2^T & D_3^T & \ldots & D_n^T \end{pmatrix} \begin{pmatrix} K_{11} & K_{12} & K_{13} & \ldots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \ldots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \ldots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \ldots & K_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \end{pmatrix}$$

$$+ \begin{pmatrix} D_1^T & D_2^T & D_3^T & \ldots & D_n^T \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} + C. \tag{4.18}$$

From the formulation of Π, the formula (1.12) can be written as a symmetric representation,

$$K_{ij} = K_{ji}^T$$

For cover i, equations

$$\frac{\partial \Pi}{\partial u_i} = 0, \qquad \frac{\partial \Pi}{\partial v_i} = 0 \tag{4.19}$$

represent the equilibrium of all the loads and contact forces acting on cover i or node i along x and y directions respectively.

The differentiations

$$\frac{\partial^2 \Pi}{\partial d_{ir} \partial d_{js}}, \quad r, s = 1, 2, \tag{4.20}$$

are the coefficients of unknowns d_{js} of the equilibrium equation (4.18) for variable d_{ir}.

The differentiations (4.20) of the total potential energy Π, produces n equations of submatrices as there are n physical covers or nodes in the manifold. The simultaneous equilibrium equations have the same coefficient matrix as the quadratic form of Π in (4.18):

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} & \dots & K_{1n} \\ K_{21} & K_{22} & K_{23} & \dots & K_{2n} \\ K_{31} & K_{32} & K_{33} & \dots & K_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & K_{n3} & \dots & K_{nn} \end{pmatrix} \begin{pmatrix} D_1 \\ D_2 \\ D_3 \\ \vdots \\ D_n \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix}. \qquad (4.21)$$

Because each node or cover has two degrees of freedom in 2-d FEM manifold, each element $[K_{ij}]$ in the coefficient matrix given by equation (4.21) is a 2×2 submatrix. $\{D_i\}$ and $\{F_i\}$ are the 2×1 submatrices.

In case the cover functions are series with m unknown coefficients, the displacement function $(u_i(x,y), v_i(x,y))$ have $2m$ unknowns. Therefore each submatrices K_{ij} in the coefficient matrix given by equation (4.21) is then a $2m \times 2m$ submatrix. D_i and F_i are $2m \times 1$ submatrices, where D_i represents the displacement variables of physical cover i.

$$\{D_i\} = \begin{pmatrix} d_{i1} \\ d_{i2} \\ d_{i3} \\ d_{i4} \\ d_{i5} \\ d_{i6} \\ \vdots \\ \vdots \\ d_{i2m-1} \\ d_{i2m} \end{pmatrix} = \begin{pmatrix} u_{i1} \\ v_{i1} \\ u_{i2} \\ v_{i2} \\ u_{i3} \\ v_{i3} \\ \vdots \\ \vdots \\ u_{im} \\ v_{im} \end{pmatrix}$$

Chapter 5

Element Matrices of Finite Element Covers

5.1 Stiffness Matrix for Finite Element Covers

For FEM, the integration domains of the stiffness matrices are whole elements with standard boundaries. For manifold method, the integration domains of the stiffness matrices are the manifold elements, which can be part of the elements.

Same as FEM method, the relationship between stress and strain, is given by

$$
\begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix} \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = [E] \begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix}, \tag{5.1}
$$

where

$$
[E] = \frac{E}{1 - \nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{pmatrix},
$$

and E, ν are the Young's modulus and Poisson's ratio respectively. And

$$
\begin{pmatrix} u \\ v \end{pmatrix} = [T_e]\{D_e\}
$$

$$
= ([T_1] \quad [T_2] \quad [T_3]) \begin{pmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \end{pmatrix},
$$

$$
\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix}
$$

where

$$
[T_i] = \begin{pmatrix} f_i & 0 \\ 0 & f_i \end{pmatrix}, \qquad \{D_i\} = \begin{pmatrix} u_i \\ v_i \end{pmatrix},
$$

and

$$
\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \begin{pmatrix} f_{11} + f_{21}x + f_{31}y \\ f_{12} + f_{22}x + f_{32}y \\ f_{13} + f_{23}x + f_{33}y \end{pmatrix}.
$$

Then

$$
\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \tau_{xy} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & 0 & \frac{\partial f_2}{\partial x} & 0 & \frac{\partial f_3}{\partial x} & 0 \\ 0 & \frac{\partial f_1}{\partial y} & 0 & \frac{\partial f_2}{\partial y} & 0 & \frac{\partial f_3}{\partial y} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial x} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} f_{21} & 0 & f_{22} & 0 & f_{23} & 0 \\ 0 & f_{31} & 0 & f_{32} & 0 & f_{33} \\ f_{31} & f_{21} & f_{32} & f_{22} & f_{33} & f_{23} \end{pmatrix} \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix}. \tag{5.2}
$$

Let

$$
[B_e] = \left([B_1] \quad [B_2] \quad [B_3] \right), \tag{5.3}
$$

where

$$
[B_i] = \begin{pmatrix} f_{2i} & 0 \\ 0 & f_{3i} \\ f_{3i} & f_{2i} \end{pmatrix}, \qquad i = 1, 2, 3. \tag{5.4}
$$

Then

$$
\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ \tau_{xy} \end{pmatrix} = [B_e]\{D_e\} = \left([B_1] \quad [B_2] \quad [B_3] \right) \begin{pmatrix} \{D_1\} \\ \{D_2\} \\ \{D_3\} \end{pmatrix}. \tag{5.5}
$$

The strain energy Π_e done by the elastic stresses of element e is

$$
\Pi_e = \iint_{\mathcal{A}} \frac{1}{2} (\epsilon_x \sigma_x + \epsilon_y \sigma_y + \gamma_{xy} \tau_{xy}) \, dx \, dy, \tag{5.6}
$$

where the integration is over the entire material area $\mathcal{A}$ in that element. Then

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$$\Pi_e = \iint_{\mathcal{A}} \frac{1}{2} \begin{pmatrix} \epsilon_x & \epsilon_y & \gamma_{xy} \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix} dx\,dy$$

$$= \frac{1}{2} \iint_{\mathcal{A}} \{D_e\}^T [B_e]^T [E][B_e]\{D_e\}\, dx\,dy$$

$$= \frac{1}{2}\{D_e\}^T \left[\iint_{\mathcal{A}} [B_e]^T [E][B_e]\, dx\,dy \right] \{D_e\}$$

$$= \frac{1}{2}\{D_e\}^T \Big(S^e [B_e]^T [E][B_e] \Big) \{D_e\}, \tag{5.7}$$

where S^e is the area of that element.

Therefore,

$$S^e [B_e]^T [E][B_e] = S^e \begin{pmatrix} [B_1]^T \\ [B_2]^T \\ [B_3]^T \end{pmatrix} [E]\Big(\begin{matrix} [B_1] & [B_2] & [B_3] \end{matrix} \Big), \tag{5.8}$$

is the element stiffness matrix.

Then

$$S^e [B_r]^T [E][B_s] \qquad \rightarrow \qquad [K_{i(r)i(s)}], \qquad r, s = 1, 2, 3,$$

and

$$[B_i] = \begin{pmatrix} f_{2i} & 0 \\ 0 & f_{3i} \\ f_{3i} & f_{2i} \end{pmatrix}, \qquad i = 1, 2, 3,$$

where

$$i(\ell) = \begin{cases} i_1, & \ell = 1, & \text{first node of the element} \\ i_2, & \ell = 2, & \text{second node of the element} \\ i_3, & \ell = 3, & \text{third node of the element} \end{cases}$$

5.2 Initial Stress Matrix for Finite Element Covers

Following the time sequence, the manifold method computes step by step. The computed stresses of previous time step will be the initial stresses of the next time step. Therefore the initial stresses are essential for manifold computation.

For the element e, the potential energy of the initial constant stresses

$$\{\sigma_e^0\} = (\sigma_x^0 \quad \sigma_y^0 \quad \tau_{xy}^0)^T$$

is

$$\Pi_\sigma = \iint_{\mathcal{A}} (\epsilon_x \sigma_x^0 + \epsilon_y \sigma_y^0 + \gamma_{xy} \tau_{xy}^0) \, dx \, dy$$

$$= \iint_{\mathcal{A}} (\epsilon_x \quad \epsilon_y \quad \gamma_{xy}) \begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} dx \, dy$$

$$= \iint_{\mathcal{A}} \{D_e\}^T [B_e]^T \{\sigma_e^0\} \, dx \, dy$$

$$= S^e \{D_e\}^T [B_e]^T \{\sigma_e^0\}, \tag{5.9}$$

where S^e is the area of that element. Therefore,

$$-S^e [B_e]^T \{\sigma_e^0\} = -S^e \begin{pmatrix} [B_1]^T \\ [B_2]^T \\ [B_3]^T \end{pmatrix} \begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} \tag{5.10}$$

is the load matrix.

Then

$$-S^e [B_r]^T \begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} \qquad \rightarrow \qquad \{F_{i(r)}\}, \qquad r = 1, 2, 3.$$

where

$$i(\ell) = \begin{cases} i_1, & \ell = 1, & \text{first node of the element} \\ i_2, & \ell = 2, & \text{second node of the element} \\ i_3, & \ell = 3, & \text{third node of the element} \end{cases}$$

5.3 Point Loading Matrix for Finite Element Covers

Different from ordinary FEM method, a load point can be any point in its element. The point loading force

$$\begin{pmatrix} F_x & F_y \end{pmatrix}^T$$

acts on point (x, y) of element e. And

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}.$$

The potential energy due to the point loading is

$$\begin{aligned}
\Pi_p &= -(F_x u + F_y v) \\
&= -\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} \\
&= -\{D_e\}^T [T_e(x, y)]^T \begin{pmatrix} F_x \\ F_y \end{pmatrix}.
\end{aligned} \qquad (5.11)$$

Therefore,

$$[T_e]^T \begin{pmatrix} F_x \\ F_y \end{pmatrix} = \begin{pmatrix} [T_1]^T \\ [T_2]^T \\ [T_3]^T \end{pmatrix} \begin{pmatrix} F_x \\ F_y \end{pmatrix} \qquad (5.12)$$

is the load matrix.

Then

$$[T_r]^T \begin{pmatrix} F_x \\ F_y \end{pmatrix} \qquad \rightarrow \qquad \{F_{i(r)}\}, \qquad r = 1, 2, 3,$$

where

$$i(\ell) = \begin{cases} i_1, & \ell = 1, \qquad \text{first node of the element} \\ i_2, & \ell = 2, \qquad \text{second node of the element} \\ i_3, & \ell = 3, \qquad \text{third node of the element} \end{cases}$$

5.4 Body Loading Matrix for Finite Element Covers

Assuming that

$$\begin{pmatrix} f_x & f_y \end{pmatrix}^T \qquad \cdot$$

is the constant body force acting on the material area of element e.

The potential energy due to the body loading is

$$\Pi_w = -\iint_{\mathcal{A}} (f_x u + f_y v)\, dx\, dy$$

$$= -\iint_{\mathcal{A}} (\,u \quad v\,) \binom{f_x}{f_y}\, dx\, dy$$

$$= -\{D_e\}^T \left[\iint_{\mathcal{A}} [T_e]^T\, dx\, dy\right] \binom{f_x}{f_y}. \tag{5.13}$$

Therefore,

$$\left[\iint_{\mathcal{A}} [T_e]^T\, dx\, dy\right] \binom{f_x}{f_y} = \left[\iint_{\mathcal{A}} \begin{pmatrix} [T_1]^T \\ [T_2]^T \\ [T_3]^T \end{pmatrix} dx\, dy\right] \binom{f_x}{f_y}. \tag{5.14}$$

is the body loading matrix. And

$$\iint_{\mathcal{A}} [T_i]^T\, dx\, dy = \iint_{\mathcal{A}} \begin{pmatrix} f_i & 0 \\ 0 & f_i \end{pmatrix} dx\, dy$$

$$= \iint_{\mathcal{A}} \begin{pmatrix} f_{1i} + f_{2i}x + f_{3i}y & 0 \\ 0 & f_{1i} + f_{2i}x + f_{3i}y \end{pmatrix} dx\, dy$$

$$= \begin{pmatrix} f_{1i}S^e + f_{2i}S^e_x + f_{3i}S^e_y & 0 \\ 0 & f_{1i}S^e + f_{2i}S^e_x + f_{3i}S^e_y \end{pmatrix}.$$

Then

$$\iint_{\mathcal{A}} [T_e]^T\, dx\, dy = \begin{pmatrix} S_i & 0 \\ 0 & S_i \end{pmatrix},$$

and

$$S_i = f_{1i}S^e + f_{2i}S^e_x + f_{3i}S^e_y,$$

where

$$S^e = \iint_{\mathcal{A}} dx\, dy, \tag{5.15}$$

$$S^e_x = \iint_{\mathcal{A}} x\, dx\, dy, \tag{5.16}$$

$$S^e_y = \iint_{\mathcal{A}} y\, dx\, dy. \tag{5.17}$$

Then

$$\begin{pmatrix} S_r & 0 \\ 0 & S_r \end{pmatrix} \begin{pmatrix} f_x \\ f_y \end{pmatrix} \qquad \rightarrow \qquad \{F_{i(r)}\}, \qquad i = 1, 2, 3.$$

where

$$i(\ell) = \begin{cases} i_1, & \ell = 1, & \text{first node of the element} \\ i_2, & \ell = 2, & \text{second node of the element} \\ i_3, & \ell = 3, & \text{third node of the element} \end{cases}$$

5.5 Inertia Force Matrix for Finite Element Covers

The inertia force matrix is equivalent to the mass matrix of FEM. This matrix is the most important matrix of manifold method. Giving a small time steps, the inertia force matrix will control the movements of all points of the whole material volume. Choosing small time steps, the discontinuous contact computation will be stable.

Considering the current time step, denote

$$\begin{pmatrix} u(t) & v(t) \end{pmatrix}^T$$

as the time dependent displacements of any point (x, y) of element e and $\mathcal{M}$ as the mass per unit area. The force of inertia per unit area is

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = -\mathcal{M} \frac{\partial^2}{\partial t^2} \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = -\mathcal{M}[T_e] \frac{\partial^2 \{D_e(t)\}}{\partial t^2}. \tag{5.18}$$

The potential energy of the inertia force of element e is

$$\begin{aligned} \Pi_i &= -\iint_A (u \quad v) \begin{pmatrix} f_x \\ f_y \end{pmatrix} dx \, dy \\ &= \iint_A \mathcal{M} (u \quad v) [T_e] \frac{\partial^2 \{D_e(t)\}}{\partial t^2} dx \, dy. \end{aligned} \tag{5.19}$$

Assume $\{D_e(0)\} = \{0\}$ as the element displacements at the beginning of the time step, $\{D_e(\Delta)\} = \{D_e\}$ as the displacements at the end of the time step, Δ as time interval of this time step. Then

$$\{D_e\} = \{D_e(\Delta)\} = \{D_e(0)\} + \Delta\frac{\partial\{D_e(0)\}}{\partial t} + \frac{\Delta^2}{2}\frac{\partial^2\{D_e(0)\}}{\partial t^2}$$

$$= \Delta\frac{\partial\{D_e(0)\}}{\partial t} + \frac{\Delta^2}{2}\frac{\partial^2\{D_e(0)\}}{\partial t^2}, \tag{5.20}$$

$$\frac{\partial^2\{D_e(0)\}}{\partial t^2} = \frac{2}{\Delta^2}\{D_e\} - \frac{2}{\Delta}\frac{\partial\{D_e(0)\}}{\partial t} = \frac{2}{\Delta^2}\{D_e\} - \frac{2}{\Delta}\{V_e(0)\}, \tag{5.21}$$

where

$$\{V_e(0)\} = \frac{\partial\{D_e(0)\}}{\partial t}, \tag{5.22}$$

is the velocity at the beginning of the time step. Then the potential energy becomes

$$\Pi_i = \mathcal{M}\{D_e\}^T \left[\iint_\mathcal{A}[T_e]^T[T_e]\,dx\,dy\right]\left(\frac{2}{\Delta^2}\{D_e\} - \frac{2}{\Delta}\{V_e(0)\}\right). \tag{5.23}$$

Therefore,

$$-\left[\iint_\mathcal{A}[T^e]^T[T_e]\,dx\,dy\right]\left(\frac{2\mathcal{M}}{\Delta^2}\{D_e\} - \frac{2\mathcal{M}}{\Delta}\{V_e(0)\}\right). \tag{5.24}$$

is the load matrix. Thus,

$$\frac{2\mathcal{M}}{\Delta^2}\left[\iint_\mathcal{A}[T_e]^T[T_e]\,dx\,dy\right]$$

$$= \frac{2\mathcal{M}}{\Delta^2}\left[\iint_\mathcal{A}\begin{pmatrix}[T_1]^T\\[T_2]^T\\[T_3]^T\end{pmatrix}([T_1]\quad[T_2]\quad[T_3])\,dx\,dy\right] \quad\rightarrow\quad [K_e], \tag{5.25}$$

and

$$\frac{2\mathcal{M}}{\Delta}\left[\iint_\mathcal{A}[T_e]^T[T_e]\,dx\,dy\right]\{V_e(0)\}$$

$$= \frac{2\mathcal{M}}{\Delta}\left[\iint_\mathcal{A}\begin{pmatrix}[T_1]^T\\[T_2]^T\\[T_3]^T\end{pmatrix}([T_1]\quad[T_2]\quad[T_3])\,dx\,dy\right]\{V_e(0)\} \quad\rightarrow\quad \{F_e\}. \tag{5.26}$$

Then

$$\frac{2\mathcal{M}}{\Delta^2}\left[\iint_{\mathcal{A}}[T_r]^T[T_s]\,dx\,dy\right] \qquad \rightarrow \qquad [K_{i(r)\,i(s)}], \qquad r,s = 1,2,3. \qquad (5.27)$$

$$\frac{2\mathcal{M}}{\Delta}\left[\iint_{\mathcal{A}}[T_r]^T[T_s]\,dx\,dy\right]\{V_s(0)\} \qquad \rightarrow \qquad \{F_{i(r)}\}, \qquad \begin{cases} r = 1,2,3, \\ s = 1,2,3, \end{cases} \quad (5.28)$$

where

$$\{V_s(0)\} = \frac{\partial}{\partial t}\begin{pmatrix} u_s(0) \\ v_s(0) \end{pmatrix}.$$

The matrix integration can be computed as

$$\iint_{\mathcal{A}}[T_r]^T[T_s]\,dx\,dy$$

$$= \iint_{\mathcal{A}}\begin{pmatrix} f_r f_s & 0 \\ 0 & f_r f_s \end{pmatrix}dx\,dy = \begin{pmatrix} t_{rs} & 0 \\ 0 & t_{rs} \end{pmatrix},$$

where

$$t_{rs} = \iint_{\mathcal{A}} f_r f_s\,dx\,dy$$

$$= \iint_{\mathcal{A}}(f_{1r} + f_{2r}x + f_{3r}y)(f_{1s} + f_{2s}x + f_{3s}y)\,dx\,dy,$$

and

$$\begin{aligned} t_{rs} =\ & f_{1r}f_{1s}S^e \\ & + (f_{1r}f_{2s} + f_{1s}f_{2r})S_x^e \\ & + (f_{1r}f_{3s} + f_{1s}f_{3r})S_y^e \\ & + f_{2r}f_{2s}S_{xx}^e \\ & + f_{3r}f_{3s}S_{yy}^e \\ & + (f_{2r}f_{3s} + f_{2s}f_{3r})S_{xy}^e \end{aligned} \qquad (5.29)$$

where

$$S_{xy}^e = \iint_{\mathcal{A}} xy\,dx\,dy, \qquad (5.30)$$

$$S_{xx}^e = \iint_{\mathcal{A}} x^2\,dx\,dy, \qquad (5.31)$$

$$S_{yy}^e = \iint_{\mathcal{A}} y^2\,dx\,dy. \qquad (5.32)$$

As this element of the manifold method is a generally shaped polygon,

$$P_1 \ P_2 \ \ldots \ P_{m-1} \ P_m \ P_1, \quad P_1 = P_{m+1}, \quad P_i = (x_i, y_i)$$

are its ordered vertices rotating from axis x to axis y. Denote $P_0 = (x_0, y_0)$ as the arbitrary chosen point, The analytical solutions of these integrations are the following:

$$S^e = \frac{1}{2} \sum_{i=1}^{m} \begin{vmatrix} 1 & x_0 & y_0 \\ 1 & x_i & y_i \\ 1 & x_{i+1} & y_{i+1} \end{vmatrix}, \tag{5.33}$$

$$S_x^e = \frac{S^e}{3} \sum_{i=1}^{m} (x_0 + x_i + x_{i+1}), \tag{5.34}$$

$$S_y^e = \frac{S^e}{3} \sum_{i=1}^{m} (y_0 + y_i + y_{i+1}), \tag{5.35}$$

$$S_{xx}^e = \frac{S^e}{6} \sum_{i=1}^{m} (x_0^2 + x_i^2 + x_{i+1}^2 + x_i x_0 + x_{i+1} x_0 + x_i x_{i+1}), \tag{5.36}$$

$$S_{yy}^e = \frac{S^e}{6} \sum_{i=1}^{m} (y_0^2 + y_i^2 + y_{i+1}^2 + y_i y_0 + y_{i+1} y_0 + y_i y_{i+1}), \tag{5.37}$$

$$S_{xy}^e = \frac{S^e}{12} \sum_{i=1}^{m} (2x_0 y_0 + 2x_i y_i + 2x_{i+1} y_{i+1}$$

$$+ x_i y_0 + x_{i+1} y_0 + x_0 y_i + x_0 y_{i+1} + x_i y_{i+1} + x_{i+1} y_i). \tag{5.38}$$

Then the final formula is

$$\frac{2\mathcal{M}}{\Delta^2} \begin{pmatrix} t_{rs} & 0 \\ 0 & t_{rs} \end{pmatrix} \quad \rightarrow \quad [K_{i(r)i(s)}], \quad r, s = 1, 2, 3, \tag{5.39}$$

$$\frac{2\mathcal{M}}{\Delta} \begin{pmatrix} t_{rs} & 0 \\ 0 & t_{rs} \end{pmatrix} \{V_s(0)\} \quad \rightarrow \quad \{F_{i(r)}\}, \quad \begin{cases} r = 1,2,3, \\ s = 1,2,3. \end{cases} \tag{5.40}$$

where

$$i(\ell) = \begin{cases} i_1, & \ell = 1, \quad \text{first node of the element} \\ i_2, & \ell = 2, \quad \text{second node of the element} \\ i_3, & \ell = 3, \quad \text{third node of the element} \end{cases}$$

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5.6 Fixed Point Matrix for Finite Element Covers

As a boundary condition, some of the elements are fixed at specific points. The constraint can be applied by using two very stiff springs. Assume the fixed point is (x, y) at element e with the displacements

$$\begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The computed displacements are

$$\begin{pmatrix} u & v \end{pmatrix}^T$$

There are two springs which are along the x and y directions respectively. The stiffness of the springs is p. The spring forces are

$$\begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} -pu \\ -pv \end{pmatrix}.$$

The strain energy of the spring is Π_f, then

$$\begin{aligned} \Pi_f &= \frac{p}{2}(u^2 + v^2) \\ &= \frac{p}{2} \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \frac{p}{2} \{D_e\}^T [T_e]^T [T_e] \{D_e\}. \end{aligned} \tag{5.41}$$

Then

$$p[T_e]^T[T_e] = p \begin{pmatrix} [T_1]^T \\ [T_2]^T \\ [T_3]^T \end{pmatrix} \begin{pmatrix} [T_1] & [T_2] & [T_3] \end{pmatrix}. \tag{5.42}$$

is the $[K_e]$ matrix. Therefore,

$$p f_r f_s \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \rightarrow \quad [K_{i(r)i(s)}], \qquad r, s = 1, 2, 3.$$

where

$$i(\ell) = \begin{cases} i_1, & \ell = 1, \quad \text{first node of the element} \\ i_2, & \ell = 2, \quad \text{second node of the element} \\ i_3, & \ell = 3, \quad \text{third node of the element} \end{cases}$$

Chapter 6

Entrance Theory on Contacts of
Finite Element Covers

6.1 Contacts and kinematics for Finite Element Covers

As the physical covers are formed by the triangle element meshes and physical boundaries, the contact surfaces are edges from

[1] block boundaries,

[2] one of two sides of joints in the material field.

The vertices of the edges can be

[1] the vertices of physical boundaries or joints,

[2] the intersection points of triangular with block boundaries or joints.

The simple penetration inequality in Chapter 3

$$\begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} < 0. \tag{6.1}$$

is fundamental for contacts.

[1] the inequality (6.1) always correct when three points P_1, P_2, P_3 moves simultaneously,

[2] the inequality (6.1) is still accurate when rotation takes place,

[3] the assumption of small displacements (u_i, v_i) is not needed,

[4] no second order infinite small is omitted.

It seems, the judgment of three point penetration is a simple matter. One can find several formulae for the same judgment. From mathematics the inequality (6.1) is the only

complete accurate formula. Neglecting second order small of (u_i, v_i), other formulae are also correct and accurate if the movements are small enough.

The formula of contact distance (3.3) of Chapter 3 is also an accurate formula.

$$d = \frac{1}{l} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} \tag{6.2}$$

$$l = \sqrt{(x_2 + u_2 - x_3 - u_3)^2 + (y_2 + v_2 - y_3 - v_3)^2}$$

However, in (6.2), formula of l is complex and should be simplified. Under assumption of small pre-existing distance

$$d = \frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Distance formula can be simplified as

$$d = \frac{1}{\sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix} \tag{6.3}$$

In formula (6.3), the denominator does not include unknown variables (u_i, v_i). Therefore, formula (6.3) is a simple function with respect to unknown variable (u_i, v_i).

The inequalities (6.1) will be a equation having "=" sign instead of "<" sign if its contact becomes a close contact. A stiff spring will be added on the contact to fulfill the equation (6.1).

The friction law or Coloumb's law is also inequalities.

$$\mathcal{F} \le \mathcal{N} \tan(\phi) + \mathcal{C} \tag{6.4}$$

The solution of this inequality is

$$\begin{cases} apply \quad a \quad shear \quad stiff \quad spring & if \mathcal{F} < \mathcal{N}\tan(\phi) + C \\ apply \quad a \quad pair \quad of \quad friction \quad forces & if \mathcal{F} = \mathcal{N}\tan(\phi) + C \end{cases}$$

In this way, the inequality of friction law is also transferred to equations.

The kinematics of the triangle elements is the same as the general kinematics in Chapter 3:

[1] distance criteria of contacts,

[2] angle criteria of contacts,

[3] definition of entrance lines,

[4] penetration judgments,

[5] penetration prevention,

[6] entrance position.

6.2 Normal Contact Matrix for Finite Element Covers

Assume P_1 is a vertex, P_2P_3 is the entrance line and (x_k, y_k) and (u_k, v_k) are the coordinates and displacement of $P_k, k = 1, 2, 3$ respectively. If points P_1, P_2 and P_3 rotate in the same sense as the rotation of ox to oy (see Figure 3.3 and 3.4), then the distance d from P_1 to entrance line P_2P_3 is

$$d = \frac{\Delta}{l} = \frac{1}{l} \begin{vmatrix} 1 & x_1 + u_1 & y_1 + v_1 \\ 1 & x_2 + u_2 & y_2 + v_2 \\ 1 & x_3 + u_3 & y_3 + v_3 \end{vmatrix}, \qquad (6.5)$$

$$l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}.$$

d should be $zero$ if P_1 passed edge $P_2\ P_3$

Let

$$S_0 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix},$$

$$\Delta \approx S_0 + \begin{vmatrix} 1 & u_1 & y_1 \\ 1 & u_2 & y_2 \\ 1 & u_3 & y_3 \end{vmatrix} + \begin{vmatrix} 1 & x_1 & v_1 \\ 1 & x_2 & v_2 \\ 1 & x_3 & v_3 \end{vmatrix}, \tag{6.6}$$

Then

$$\begin{aligned} \Delta = S_0 &+ \left((y_2 - y_3) \quad (x_3 - x_2) \right) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \\ &+ \left((y_3 - y_1) \quad (x_1 - x_3) \right) \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \\ &+ \left((y_1 - y_2) \quad (x_2 - x_1) \right) \begin{pmatrix} u_3 \\ v_3 \end{pmatrix}. \end{aligned} \tag{6.7}$$

And

$$\begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = [T_i(x_1, y_1)]\{D_i\},$$

$$\begin{pmatrix} u_2 \\ v_2 \end{pmatrix} = [T_j(x_2, y_2)]\{D_j\},$$

$$\begin{pmatrix} u_3 \\ v_3 \end{pmatrix} = [T_j(x_3, y_3)]\{D_j\},$$

then

$$\begin{aligned} \Delta = S_0 &+ \left((y_2 - y_3) \quad (x_3 - x_2) \right) [T_i(x_1, y_1)]\{D_i\}, \\ &+ \left((y_3 - y_1) \quad (x_1 - x_3) \right) [T_j(x_2, y_2)]\{D_j\}, \\ &+ \left((y_1 - y_2) \quad (x_2 - x_1) \right) [T_j(x_3, y_3)]\{D_j\}. \end{aligned} \tag{6.8}$$

Denote

$$\{H\} = \frac{1}{l}[T_i(x_1, y_1)]^T \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix}, \tag{6.9}$$

$$\begin{aligned} \{G\} = &\frac{1}{l}[T_j(x_2, y_2)]^T \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix} \\ &+ \frac{1}{l}[T_j(x_3, y_3)]^T \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix}, \end{aligned} \tag{6.10}$$

then

$$\begin{aligned} \{H\} &= \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix} \\ &= \frac{1}{l} \begin{pmatrix} [T_1]^T(x_1, y_1) \\ [T_2]^T(x_1, y_1) \\ [T_3]^T(x_1, y_1) \end{pmatrix} \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix}, \end{aligned}$$

$$\{G\} = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix}$$

$$= \frac{1}{l} \begin{pmatrix} [T_1]^T(x_2, y_2) \\ [T_2]^T(x_2, y_2) \\ [T_3]^T(x_2, y_2) \end{pmatrix}^T \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix}$$

$$+ \frac{1}{l} \begin{pmatrix} [T_1]^T(x_3, y_3) \\ [T_2]^T(x_3, y_3) \\ [T_3]^T(x_3, y_3) \end{pmatrix}^T \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix},$$

where

$$\{H_r\} = \frac{1}{l} \begin{pmatrix} f_r(x_1,\ y_1) & 0 \\ 0 & f_r(x_1,\ y_1) \end{pmatrix} \begin{pmatrix} y_2 - y_3 \\ x_3 - x_2 \end{pmatrix},$$

$$\{G_r\} = \frac{1}{l} \begin{pmatrix} f_r(x_2,\ y_2) & 0 \\ 0 & f_r(x_2,\ y_2) \end{pmatrix} \begin{pmatrix} y_3 - y_1 \\ x_1 - x_3 \end{pmatrix}$$

$$+ \frac{1}{l} \begin{pmatrix} f_r(x_3,\ y_3) & 0 \\ 0 & f_r(x_3,\ y_3) \end{pmatrix} \begin{pmatrix} y_1 - y_2 \\ x_2 - x_1 \end{pmatrix},$$

and

$$f_r(x_1,\ y_1) = f_{1r} + f_{2r}x_1 + f_{3r}y_1 \qquad \text{at element } i,$$

$$f_r(x_2,\ y_2) = f_{1r} + f_{2r}x_2 + f_{3r}y_2 \qquad \text{at element } j,$$

$$f_r(x_3,\ y_3) = f_{1r} + f_{2r}x_3 + f_{3r}y_3 \qquad \text{at element } j.$$

Then (6.5) becomes

$$d = \{H\}^T\{D_i\} + \{G\}^T\{D_j\} + \frac{S_0}{l}. \tag{6.11}$$

The potential energy of the normal spring is

$$\Pi_s = \frac{p}{2}d^2$$

$$= \frac{p}{2} \left(\{H\}^T\{D_i\} + \{G\}^T\{D_j\} + \frac{S_0}{l} \right)^2$$

$$= \frac{p}{2} \Big[\{D_i\}^T\{H\}\{H\}^T\{D_i\} + \{D_j\}^T\{G\}\{G\}^T\{D_j\} + 2\{D_i\}^T\{H\}\{G\}^T\{D_j\}$$

$$+ 2\left(\frac{S_0}{l}\right)\{D_i\}^T\{H\} + 2\left(\frac{S_0}{l}\right)\{D_j\}^T\{G\} + \left(\frac{S_0}{l}\right)^2 \Big]. \tag{6.12}$$

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Thus,

$$p\{H\}\{H\}^T = \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix} \left(\{H_1\}^T \quad \{H_2\}^T \quad \{H_3\}^T \right), \qquad (6.13)$$

$$p\{H\}\{G\}^T = \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix} \left(\{G_1\}^T \quad \{G_2\}^T \quad \{G_3\}^T \right), \qquad (6.14)$$

$$p\{G\}\{H\}^T = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix} \left(\{H_1\}^T \quad \{H_2\}^T \quad \{H_3\}^T \right), \qquad (6.15)$$

$$p\{G\}\{G\}^T = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix} \left(\{G_1\}^T \quad \{G_2\}^T \quad \{G_3\}^T \right), \qquad (6.16)$$

are the $[K_e]$ matrices; and

$$-p\left(\frac{S_0}{l}\right)\{H\} = -p\left(\frac{S_0}{l}\right) \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix}, \qquad (6.17)$$

$$-p\left(\frac{S_0}{l}\right)\{G\} = -p\left(\frac{S_0}{l}\right) \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix}, \qquad (6.18)$$

are the load matrices.

Then

$$p\{H_r\}\{H_s\}^T \quad \rightarrow \quad [K_{i(r)\,i(s)}], \qquad r,s = 1,2,3,$$

$$p\{H_r\}\{G_s\}^T \quad \rightarrow \quad [K_{i(r)\,j(s)}], \qquad r,s = 1,2,3,$$

$$p\{G_r\}\{H_s\}^T \quad \rightarrow \quad [K_{j(r)\,i(s)}], \qquad r,s = 1,2,3,$$

$$p\{G_r\}\{G_s\}^T \quad \rightarrow \quad [K_{j(r)\,j(s)}], \qquad r,s = 1,2,3,$$

$$-p\left(\frac{S_0}{l}\right)\{H_r\} \quad \rightarrow \quad \{F_{i(r)}\}, \qquad r = 1,2,3,$$

$$-p\left(\frac{S_0}{l}\right)\{G_r\} \quad \rightarrow \quad \{F_{j(r)}\}, \qquad r = 1,2,3,$$

where

$$\text{for element } i \qquad \begin{cases} i(1) = i_1, \\ i(2) = i_2, \\ i(3) = i_3, \end{cases}$$

and

$$\text{for element } j \qquad \begin{cases} j(1) = j_1, \\ j(2) = j_2, \\ j(3) = j_3. \end{cases}$$

6.3 Shear Contact Matrix for Finite Element Covers

In Figure 3.8, (x_0, y_0) is on the edge $P_2 P_3$ and is the assumed contact point of vertex P_1. The shear spring is on the direction $P_2 P_3$, and connects vertices P_1 and P_0. Let

$$l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2},$$

the shear displacement of P_0 and P_1 along line $P_2 P_3$ is

$$d = \frac{1}{l} \overrightarrow{P_0 P_1} \cdot \overrightarrow{P_2 P_3} \tag{6.19}$$

$$= \frac{1}{l} \left((x_1 + u_1) - (x_0 + u_0) \quad (y_1 + v_1) - (y_0 + v_0) \right) \begin{pmatrix} (x_3 + u_3) - (x_2 + u_2) \\ (y_3 + v_3) - (y_2 + v_2) \end{pmatrix}.$$

Denote

$$S_0 = \left(x_1 - x_0 \quad y_1 - y_0 \right) \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}, \tag{6.20}$$

since $P_0 P_1$ is small,

$$d \approx \frac{S_0}{l} + \frac{1}{l} \left(x_3 - x_2 \quad y_3 - y_2 \right) \begin{pmatrix} u_1 - u_0 \\ v_1 - v_0 \end{pmatrix},$$

$$d \approx \frac{S_0}{l} + \frac{1}{l} \left(x_3 - x_2 \quad y_3 - y_2 \right) \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$$

$$+ \frac{1}{l} \left(x_2 - x_3 \quad y_2 - y_3 \right) \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{6.21}$$

Denote

$$\{H\} = \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix} = \frac{1}{l} [T_i(x_1, y_1)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}, \tag{6.22}$$

$$\{G\} = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix} = \frac{1}{l} [T_j(x_0, y_0)]^T \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix}, \tag{6.23}$$

where

$$\{H_r\} = \frac{1}{l} \begin{pmatrix} f_r(x_1, \ y_1) & 0 \\ 0 & f_r(x_1, \ y_1) \end{pmatrix} \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},$$

$$\{G_r\} = \frac{1}{l} \begin{pmatrix} f_r(x_0, \ y_0) & 0 \\ 0 & f_r(x_0, \ y_0) \end{pmatrix} \begin{pmatrix} x_2 - x_3 \\ y_2 - y_3 \end{pmatrix}$$

and

$$f_r(x_1, \ y_1) = f_{1r} + f_{2r}x_1 + f_{3r}y_1 \qquad \text{at element} i,$$

$$f_r(x_0, \ y_0) = f_{1r} + f_{2r}x_0 + f_{3r}0_2 \qquad \text{at element} j,$$

Then (6.21) becomes

$$d = \{H\}^T\{D_i\} + \{G\}^T\{D_j\} + \frac{S_0}{l}. \tag{6.24}$$

The potential energy of the shear spring is

$$\Pi_s = \frac{p}{2}d^2$$

$$= \frac{p}{2}\left(\{H\}^T\{D_i\} + \{G\}^T\{D_j\} + \frac{S_0}{l}\right)^2$$

$$= \frac{p}{2}\Big[\{D_i\}^T\{H\}\{H\}^T\{D_i\} + \{D_j\}^T\{G\}\{G\}^T\{D_j\} + 2\{D_i\}^T\{H\}\{G\}^T\{D_j\}$$

$$+ 2\left(\frac{S_0}{l}\right)\{D_i\}^T\{H\} + 2\left(\frac{S_0}{l}\right)\{D_j\}^T\{G\} + \left(\frac{S_0}{l}\right)^2\Big]. \tag{6.25}$$

Thus,

$$p\{H\}\{H\}^T = \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix}\left(\{H_1\}^T \quad \{H_2\}^T \quad \{H_3\}^T\right), \tag{6.26}$$

$$p\{H\}\{G\}^T = \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix}\left(\{G_1\}^T \quad \{G_2\}^T \quad \{G_3\}^T\right), \tag{6.27}$$

$$p\{G\}\{H\}^T = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix}\left(\{H_1\}^T \quad \{H_2\}^T \quad \{H_3\}^T\right), \tag{6.28}$$

$$p\{G\}\{G\}^T = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix}\left(\{G_1\}^T \quad \{G_2\}^T \quad \{G_3\}^T\right), \tag{6.29}$$

are the $[K_e]$ matrices, and

$$-p\left(\frac{S_0}{l}\right)\{H\} = -p\left(\frac{S_0}{l}\right)\begin{pmatrix}\{H_1\}\\\{H_2\}\\\{H_3\}\end{pmatrix}, \qquad (6.30)$$

$$-p\left(\frac{S_0}{l}\right)\{G\} = -p\left(\frac{S_0}{l}\right)\begin{pmatrix}\{G_1\}\\\{G_2\}\\\{G_3\}\end{pmatrix}, \qquad (6.31)$$

are the load matrices. Then

$$p\{H_r\}\{H_s\}^T \quad\rightarrow\quad [K_{i(r)i(s)}], \qquad r,s = 1,2,3,$$

$$p\{H_r\}\{G_s\}^T \quad\rightarrow\quad [K_{i(r)j(s)}], \qquad r,s = 1,2,3,$$

$$p\{G_r\}\{H_s\}^T \quad\rightarrow\quad [K_{j(r)i(s)}], \qquad r,s = 1,2,3,$$

$$p\{G_r\}\{G_s\}^T \quad\rightarrow\quad [K_{j(r)j(s)}], \qquad r,s = 1,2,3,$$

$$-p\left(\frac{S_0}{l}\right)\{H_r\} \quad\rightarrow\quad \{F_{i(r)}\}, \qquad r = 1,2,3,$$

$$-p\left(\frac{S_0}{l}\right)\{G_r\} \quad\rightarrow\quad \{F_{j(r)}\}, \qquad r = 1,2,3,$$

where

$$\text{for element } i \qquad \begin{cases} i(1) = i_1, \\ i(2) = i_2, \\ i(3) = i_3, \end{cases}$$

and

$$\text{for element } j \qquad \begin{cases} j(1) = j_1, \\ j(2) = j_2, \\ j(3) = j_3. \end{cases}$$

6.4 Friction Force Matrix for Finite Element Covers

When the Coulomb's law allows sliding between two sides of the boundary contacts, there exists the friction forces in the two sliding sides if the friction angle ϕ is not zero.

P_1 is on element i and P_2, P_3, P_0 are on element j. The friction force is calculated from the normal contact compressive force and the direction of the friction force is depending on the movement of P_1 relative to P_0 in the direction from P_2 to P_3. Let p be the stiffness of the normal contact spring, then the friction force

$$\mathcal{F} = p \cdot d \cdot s \cdot \tan(\phi), \tag{6.32}$$

where

d is the normal penetration distance,

$\tan(\phi)$ is the friction coefficient,

$s = sgn$ (movement of P_1 relative to P_0 in the direction from P_2 to P_3) , and

$$sgn(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0, \\ -1, & \text{if } x < 0. \end{cases}$$

The friction force $\mathcal{F}$ is along the direction

$$\frac{1}{l}\Big((x_3 - x_2) \quad (y_3 - y_2) \Big),$$

and

$$l = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$$

is the length of line $\overline{P_2 P_3}$.

Then the potential energy of friction force $\mathcal{F}$ at P_1 on element i is

$$\begin{aligned} \Pi_f &= \frac{\mathcal{F}}{l} \begin{pmatrix} u_1 & v_1 \end{pmatrix} \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix} \\ &= \mathcal{F}\{D_i\}^T [T_i(x_1, y_1)]^T \begin{pmatrix} (x_3 - x_2)/l \\ (y_3 - y_2)/l \end{pmatrix} \\ &= \mathcal{F}\{D_i\}^T \{H\}, \end{aligned} \tag{6.33}$$

where

$$\{H\} = \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix} = \frac{1}{l}[T_i(x_1, y_1)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}.$$

Then

$$-\mathcal{F}\{H\} = -\mathcal{F} \begin{pmatrix} \{H_1\} \\ \{H_2\} \\ \{H_3\} \end{pmatrix} \tag{6.34}$$

is the loading matrix for element i; and

$$-\mathcal{F}\{H_r\} \qquad \rightarrow \qquad \{F_{i(r)}\}, \qquad r = 1, 2, 3,$$

where

$$\{H_r\} = \frac{1}{l} \begin{pmatrix} f_r(x_1 \; y_1) & 0 \\ 0 & f_r(x_1 \; y_1) \end{pmatrix} \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},$$

and

$$f_r(x_1 \; y_1) = f_{1r} + f_{2r}x_1 + f_{3r}y_1.$$

Similarly, the potential energy of friction force $\mathcal{F}$ at P_0 on element j is

$$\begin{aligned}
\Pi_f &= -\frac{\mathcal{F}}{l} \begin{pmatrix} u_0 & v_0 \end{pmatrix} \begin{pmatrix} (x_3 - x_2) \\ (y_3 - y_2) \end{pmatrix} \\
&= -\mathcal{F}\{D_j\}^T [T_j(x_0, y_0)]^T \begin{pmatrix} (x_3 - x_2)/l \\ (y_3 - y_2)/l \end{pmatrix} \\
&= -\mathcal{F}\{D_j\}^T \{G\},
\end{aligned} \qquad (6.35)$$

where

$$\{G\} = \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix} = \frac{1}{l} [T_j(x_0, y_0)]^T \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix}$$

Then

$$+\mathcal{F}\{G\} = +\mathcal{F} \begin{pmatrix} \{G_1\} \\ \{G_2\} \\ \{G_3\} \end{pmatrix} \qquad (6.36)$$

is the loading matrix for element j; and

$$+\mathcal{F}\{G_r\} \qquad \rightarrow \qquad \{F_{j(r)}\}, \qquad r = 1, 2, 3,$$

where

$$\{G_r\} = \frac{1}{l} \begin{pmatrix} f_r(x_0 \; y_0) & 0 \\ 0 & f_r(x_0 \; y_0) \end{pmatrix} \begin{pmatrix} x_3 - x_2 \\ y_3 - y_2 \end{pmatrix},$$

and

$$f_r(x_0\ y_0) = f_{1r} + f_{2r}x_0 + f_{3r}y_0.$$

6.5 Spring stiffness

The stiffness p of the contact springs is important. If p is too small the penetration distances become too large so that

[1] the closed contact can not be transferred to next step;

[2] the stresses in materials can be reduced;

[3] deformation along joints and boundaries can be wrong.

If spring stiffness p is too large, the simultaneous equation can be nearly linear dependent or ill conditioning so that

[1] the solution errors can be unacceptable;

[2] the iteration method may not converge;

[3] contact displacement may not be correct.

To estimate p, the relationship of spring stiffness p and material Young's modulus E is computed. Consider a rectangular block supported by two springs as shown by Figure 6.1,

the block stress is

$$\sigma = \frac{2F}{a}$$

$$\epsilon = \frac{\sigma}{E} = \frac{2F}{aE}$$

$$d_b = b\epsilon = \frac{2bF}{aE}$$

$$d_s = \frac{F}{p}$$

$$d_b = d_s \frac{2bp}{aE} \tag{6.36}$$

where

a is the length of the rectangle

b is the height of the rectangle

F is the value of two vertical point loads

σ is the vertical stress of the rectangle

ϵ is the vertical strain of the rectangle

d_b is the vertical displacement on the top of rectangle

d_s is the vertical penetration of the spring

Assuming $a = b$, from (6.36),

$$d_b = d_s \frac{2p}{E} \tag{6.37}$$

It would be reasonable if

$$p = 20E \quad to \quad p = 100E$$

is chosen. From equation (6.37), the block displacement d_b is 40 times to 200 times of spring displacement d_s. Therefore the spring displacements are neglected.

In the current program, $p = 40E$ is chosen. There are several ways of choosing p

[1] directly enter the p value;

[2] define p in terms of E, such as let $p = 40E$;

[3] define p by the maximum spring displacement.

6.6 Non-penetration Physical Springs

The options of non-penetration springs are available only for the latest discontinuous deformation analysis code. The algorithm of non-penetration spring is still in testing stage. This algorithm hopefully will be transferred to the manifold method.

For the real materials, the contact stiffness is not as large as the program had set, but the penetration still keeps small. The algorithm of the low stiffness springs with small penetrations will make the contact more real or more physical.

The previous springs are mathematical; while the new springs are physical. The advantages of the physical springs are in the following:

[1] the spring penetrations are small so that the contact can be transferred to next step without missing;

[2] the springs can have the physical contact stiffness;

[3] contact springs with low stiffness will form better-conditioned simultaneous equations;

[4] low spring stiffness leads faster converge of iterative solution;

[5] no extra unknowns are added;

[6] submatrix structure of the equations are reserved.

Chapter 7

Two Dimensional Simplex Integrations

7.1 Simplex Integration on a Simplex

For the manifold method, the manifold elements are the integration zones which have general shapes, therefore the integrations are more difficult than the integrations of FEM. However analytical solutions were found for many cases of manifold method. The finite element method computes the integrations of complex functions in simple domains; the manifold method computes the integrations of simple functions in complex domains. The integrations on complex domain can be reduced to the integration on the simplex. The simplex is the oriented simplest domain.

The simplex has the most simple shape in $1, 2, 3, \ldots, n$ dimensional space. Different from the ordinary integration, the simplex integration has only the simplex as the integral domain. The simplex also has positive or negative orientations. Positive or negative orientations define positive or negative volumes respectively. Figure 7.1 shows the 0, 1, 2, 3 dimensional simplex.
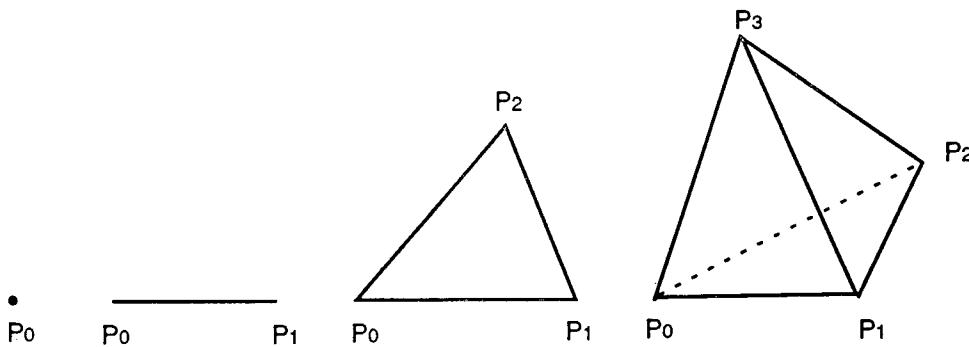
Figure 7.1. 0,1,2,3 dimensional implex

The 0 dimensional simplex is a point P_0, its volume is

$$\frac{1}{0!} \mid 1 \mid = 1 \tag{7.1}$$

The 0-d simplex integration can be considered as a normal real number.

The one dimensional simplex $P_0 P_1$ is an oriented segment, its volume is

$$\frac{1}{1!} \begin{vmatrix} 1 & x_{10} \\ 1 & x_{11} \end{vmatrix} = x_{11} - x_{10} \tag{7.2}$$

The volume of the simplex $P_1 P_0$ is the negative volume of the simplex $P_0 P_1$. The ordinary 1-d integrations are the simplex integrations:

$$\int_{P_0}^{P_1} f(x)dx = -\int_{P_1}^{P_0} f(x)dx$$

Therefore, for any co-line point P_2,

$$\int_{P_0}^{P_2} f(x)dx + \int_{P_2}^{P_1} f(x)dx = \int_{P_0}^{P_1} f(x)dx + \int_{P_1}^{P_2} f(x)dx + \int_{P_2}^{P_1} f(x)dx = \int_{P_0}^{P_1} f(x)dx$$

The 1-d integration addition is the same as vector addition. The integrations on negative vectors and positive vectors can be nullified.

Two dimensional simplex $P_0 P_1 P_2$ is an oriented triangle, its volume is

$$\frac{1}{2!} \begin{vmatrix} 1 & x_{10} & x_{20} \\ 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \end{vmatrix} \tag{7.3}$$

The volume of simplex $P_1 P_0 P_2$ is the negative volume of simplex $P_0 P_1 P_2$.

7.2 Simplex Integration on a General Polygon

Unfortunately, the two dimensional ordinary integrations are volume integration, where the volume is always positive. For the 2 dimensional ordinary integration, the integration domains have no orientations, therefore the integrations have no algebraic addition

on oriented domains. It is then necessary to introduce simplex integration on a simplex, which is a simple oriented domain.

The simplex integration is not intended to do integrations on a simplex only. Obviously a complex shape always can be subdivided into simplex, so simplex integration can be computed in each simplex, the summation of simplex integrations is the ordinary integration over the complex shape. However this is also not the way of using simplex integration.

Giving a polygon $P_1 P_2 P_2 P_4 P_5 P_6$ with $P_6 = P_1$, such that P_i rotates at the same direction as from ox to oy. For any point P_0, the algebraic addition of the 2-d simplex volume (area) $P_0 P_1 P_2$, $P_0 P_2 P_3$, $P_0 P_3 P_4$, $P_0 P_4 P_5$ and $P_0 P_5 P_1$ is the area A of the polygon. Let $P_0 = (0,0)$,

$$ A = \frac{1}{2} \sum_{i=1}^{5} \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_{1\,i} & x_{2\,i} \\ 1 & x_{1\,i+1} & x_{2\,i+1} \end{vmatrix} = \frac{1}{2} \sum_{i=1}^{5} \begin{vmatrix} x_{1\,i} & x_{2\,i} \\ x_{1\,i+1} & x_{2\,i+1} \end{vmatrix} \tag{7.4} $$

Figure 7.2 shows the addition of plus and minus areas of simplex. In Figure 7.2, the area of simplex $P_0 P_2 P_3$, $P_0 P_3 P_4$, $P_0 P_4 P_5$ and $P_0 P_5 P_1$ are positive; the area of simplex $P_0 P_1 P_2$ is negative. The algebraic sum is exactly the area of polygon $P_1 P_2 P_2 P_4 P_5 P_6$. The area A is then represented by the coordinates of boundary vertices.
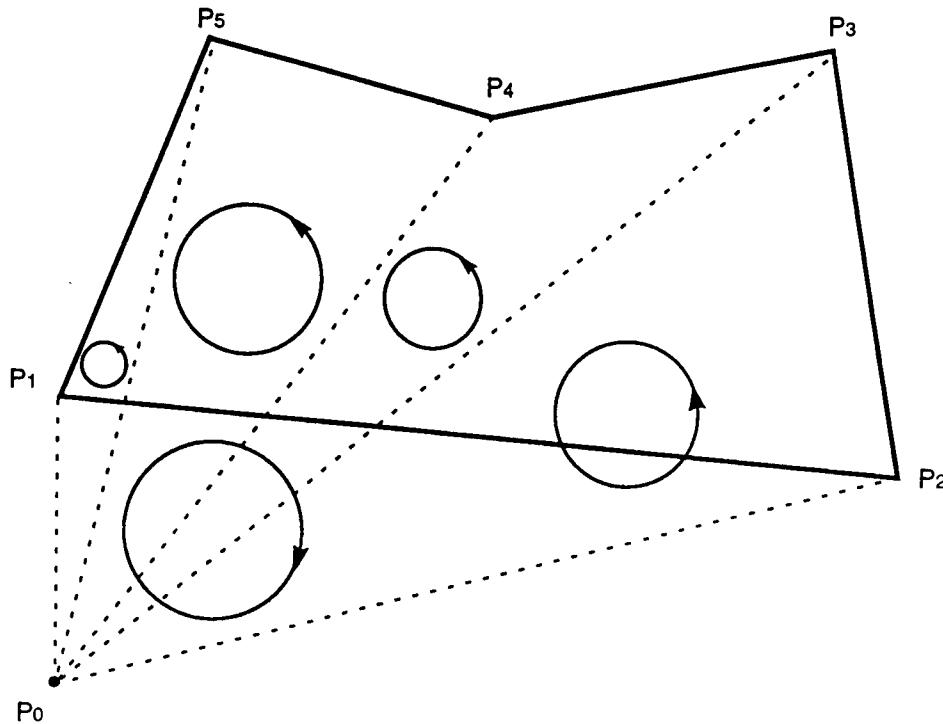

Figure 7.2. Addition of plus or minus area of simplex

In general, the simplex integration can compute ordinary integrations without subdividing 2-d domains to triangles. FEM mesh is unnecessary for simplex integration. Using simplex integration, the integration of any polynomials can be represented by the coordinates of boundary vertices of generally shaped polyhedron.

7.3 Two Dimensional Simplex Integrations

The two dimensional integrations here in this section are used for two dimensional manifold method algorithm. Since the displacement function is linear function of coordinates (x, y), the integrands are of degree 0, 1, 2.

A 2 dimensional simplex has 3 vertices

$$P_0, P_1, P_2.$$

$$
\begin{array}{ll}
P_0: & (x_{10},\quad x_{20}) \\
P_1: & (x_{11},\quad x_{21}) \\
P_2: & (x_{12},\quad x_{22})
\end{array}
\tag{7.5}
$$

The 2 dimensional coordinate simplex has 3 vertices

$$U_0, U_1, U_2.$$

$$
\begin{array}{ll}
U_0: & (\quad 0,\quad 0) \\
U_1: & (\quad 1,\quad 0) \\
U_2: & (\quad 0,\quad 1)
\end{array}
\tag{7.6}
$$

The following coordinate transformation

$$(u_1, u_2) \rightarrow (x_1, x_2)$$

transfers coordinate simplex

$$U_0 U_1 U_2$$

to normal simplex

$$P_0 P_1 P_2$$

$$
\begin{aligned}
x_1 &= x_{10} \left(1 - \sum_1^2 u_i\right) + x_{11} \quad u_1 \quad + \quad x_{12} \quad u_2 \\
x_2 &= x_{20} \left(1 - \sum_1^2 u_i\right) + x_{21} \quad u_1 \quad + \quad x_{22} \quad u_2
\end{aligned}
\tag{7.7}
$$

The Jacobi determinant is

$$
\begin{aligned}
J &= \frac{D(x_1,\ x_2)}{D(u_1,\ u_2)} \\
&= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix} \\
&= \begin{vmatrix} x_{11} - x_{10} & x_{12} - x_{10} \\ x_{21} - x_{20} & x_{22} - x_{20} \end{vmatrix} \\
&= \begin{vmatrix} 1 & x_{10} & x_{20} \\ 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \end{vmatrix}
\end{aligned}
\tag{7.8}
$$

Since $P_0 P_1 P_2$ is a 2-dimensional simplex with non-zero volume, J is non-zero. Translation can be rewritten as

$$(u_0, u_1, u_2) \to (1, x_1, x_2)$$

$$
\begin{aligned}
1 &= & u_0 &+ & u_1 &+ & u_2 \\
x_1 &= x_{10} & u_0 &+ x_{11} & u_1 &+ x_{12} & u_2 \\
x_2 &= x_{20} & u_0 &+ x_{21} & u_1 &+ x_{22} & u_2
\end{aligned}
\tag{7.9}
$$

Two dimensional simplex integration

$$S_{P_0 P_1 P_2}(m_1, m_2)$$

on a 2 dimensional simplex

$$P_0 P_1 P_2$$

is defined as normal integration times the sign of determinant J.

$$S_{P_0 P_1 P_2}(m_1, m_2)$$

$$= sign(J) \int \int_{P_0 P_1 P_2} x_1^{m_1} x_2^{m_2} \, dx_1 dx_2$$

$$= sign(J) \int \int_{U_0 U_1 U_2} x_1^{m_1} x_2^{m_2} |J| du_1 du_2$$

$$= J \int \int_{U_0 U_1 U_2} x_1^{m_1} x_2^{m_2} du_1 du_2 \tag{7.10}$$

Here

$$x_l^{m_l} = \left(\sum_{k=0}^{2} x_{lk} u_k \right)^{m_l}$$

then the two dimensional simplex integration can be represented by the following basic forms of simplex integration:

$$S_1 = \int \int_{U_0 U_1 U_2} u_0^{i_0} u_1^{i_1} u_2^{i_2} du_1 du_2$$

$$= \int \int_{u_0, u_1, u_2 \geq 0}^{u_0 + u_1 + u_2 = 1} u_0^{i_0} u_1^{i_1} u_2^{i_2} du_1 du_2$$

$$= \int_0^1 \int_0^{1 - u_2} u_2^{i_2} u_1^{i_1} u_0^{i_0} du_1 du_2$$

$$= \int_0^1 u_2^{i_2} \left(\int_0^{1 - u_2} u_1^{i_1} (1 - u_2 - u_1)^{i_0} du_1 \right) du_2 \tag{7.11}$$

After i_1 times of integration by parts, the inner integration can be computed.

$$\int_0^{1 - u_2} u_1^{i_1} (1 - u_2 - u_1)^{i_0} du_1$$

$$= \int_0^{1 - u_2} d(\frac{1}{i_1 + 1} u_1^{i_1 + 1})(1 - u_2 - u_1)^{i_0}$$

$$= \frac{1}{i_1 + 1} u_1^{i_1 + 1} (1 - u_2 - u_1)^{i_0} \Big|_0^{1 - u_2}$$

$$- \int_0^{1 - u_2} \frac{1}{i_1 + 1} u_1^{i_1 + 1} d((1 - u_2 - u_1)^{i_0})$$

$$= - \int_0^{1 - u_2} \frac{1}{i_1 + 1} u_1^{i_1 + 1} d((1 - u_2 - u_1)^{i_0})$$

$$= \int_0^{1-u_2} \frac{i_0}{i_1+1} u_1^{i_1+1} (1 - u_2 - u_1)^{i_0-1} du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0-1)}{(i_1+1)(i_1+2)} u_1^{i_1+2} (1 - u_2 - u_1)^{i_0-2} du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0-1)(i_0-2)}{(i_1+1)(i_1+2)(i_1+3)} u_1^{i_1+3} (1 - u_2 - u_1)^{i_0-3} du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0-1)(i_0-2)\ldots 2}{(i_1+1)(i_1+2)(i_1+3)\ldots(i_1+i_0-1)} u_1^{i_1+i_0-1}$$
$$(1 - u_2 - u_1)^1 du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0-1)(i_0-2)\ldots 1}{(i_1+1)(i_1+2)(i_1+3)\ldots(i_1+i_0)} u_1^{i_1+i_0} du_1$$

$$= \int_0^{1-u_2} \frac{i_0! i_1!}{(i_1+i_0)!} u_1^{i_1+i_0} du_1$$

$$= \int_0^{1-u_2} \frac{i_0! i_1!}{(i_1+i_0+1)!} d(u_1^{i_1+i_0+1})$$

$$= \frac{i_0! i_1!}{(i_0+i_1+1)!} u_1^{i_0+i_1+1} \Big|_0^{1-u_2}$$

$$= \frac{i_0! i_1!}{(i_1+i_0+1)!} (1 - u_2)^{i_1+i_0+1} \tag{7.12}$$

$$S_1 = \int_0^1 u_2^{i_2} du_2 \int_0^{1-u_2} u_1^{i_1} (1 - u_2 - u_1)^{i_0} du_1$$

$$= \int_0^1 u_2^{i_2} (1 - u_2)^{i_0+i_1+1} du_2 \frac{i_0! i_1!}{(i_0+i_1+1)!}$$

$$= \frac{i_0! i_1! i_2!}{(i_0+i_1+i_2+2)!} \tag{7.13}$$

Based on the previous formula of S_1, the two dimensional integration of polynomials of degree 0, 1, 2 can be computed.

$$S_{P_0 P_1 P_2}(0,0) = sign(J) \int \int_{P_0 P_1 P_2} dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} du_1 du_2$$

$$S_{P_0 P_1 P_2}(0,0) = sign(J) \int \int_{P_0 P_1 P_2} dx_1 dx_2$$

$$= J\frac{0!}{(0+2)!} = \frac{1}{2}J \tag{7.14}$$

$$S_{P_0P_1P_2}(1,0) = sign(J)\int\int_{P_0P_1P_2} x_1 dx_1 dx_2$$

$$= J\int\int_{U_0U_1U_2} (x_{10}u_0 + x_{11}u_1 + x_{12}u_2)du_1 du_2$$

$$= J\frac{1!}{(1+2)!}(x_{10} + x_{11} + x_{12})$$

$$S_{P_0P_1P_2}(1,0) = sign(J)\int\int_{P_0P_1P_2} x_1 dx_1 dx_2$$

$$= \frac{1}{6}J(x_{10} + x_{11} + x_{12})$$

$$S_{P_0P_1P_2}(0,1) = sign(J)\int\int_{P_0P_1P_2} x_2 dx_1 dx_2$$

$$= \frac{1}{6}J(x_{20} + x_{21} + x_{22}) \tag{7.15}$$

$$S_{P_0P_1P_2}(2,0) = sign(J)\int\int_{P_0P_1P_2} x_1^2 dx_1 dx_2$$

$$= J\int\int_{U_0U_1U_2} (x_{10}u_0 + x_{11}u_1 + x_{12}u_2)^2 du_1 du_2$$

$$= J\frac{2!}{(2+2)!}(x_{10}^2 + x_{11}^2 + x_{12}^2)$$

$$+ J\frac{1!}{(2+2)!}$$

$$\begin{array}{lll}(0 & +x_{10}x_{11} & +x_{10}x_{12}\\ +x_{11}x_{10} & +0 & +x_{11}x_{12}\\ +x_{12}x_{10} & +x_{12}x_{11} & +0)\end{array}$$

$$S_{P_0P_1P_2}(2,0) = sign(J)\int\int_{P_0P_1P_2} x_1^2 dx_1 dx_2$$

$$= J\frac{1}{24}$$

$$\begin{array}{lll}(2x_{10}x_{10} & +x_{10}x_{11} & +x_{10}x_{12}\\ +x_{11}x_{10} & +2x_{11}x_{11} & +x_{11}x_{12}\\ +x_{12}x_{10} & +x_{12}x_{11} & +2x_{12}x_{12})\end{array}$$

$$S_{P_0P_1P_2}(0,2) = sign(J)\int\int_{P_0P_1P_2} x_2^2 dx_1 dx_2$$

$$= J\frac{1}{24}$$

$$\begin{array}{lll}(2x_{20}x_{20} & +x_{20}x_{21} & +x_{20}x_{22}\\ +x_{21}x_{20} & +2x_{21}x_{21} & +x_{21}x_{22}\\ +x_{22}x_{20} & +x_{22}x_{21} & +2x_{22}x_{22})\end{array} \tag{7.16}$$

$$S_{P_0 P_1 P_2}(1,1) = sign(J) \int \int_{P_0 P_1 P_2} x_1 x_2 \, dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} (x_{10} u_0 + x_{11} u_1 + x_{12} u_2)$$

$$(x_{20} u_0 + x_{21} u_1 + x_{22} u_2) du_1 du_2$$

$$= J \frac{2!}{(2+2)!} (x_{10} x_{20} + x_{11} x_{21} + x_{12} x_{22})$$

$$+ J \frac{1!}{(2+2)!}$$

$$\begin{array}{ccc} (0 & +x_{10} x_{21} & +x_{10} x_{22} \\ +x_{11} x_{20} & +0 & +x_{11} x_{22} \\ +x_{12} x_{20} & +x_{12} x_{21} & +0) \end{array}$$

$$S_{P_0 P_1 P_2}(1,1) = sign(J) \int \int_{P_0 P_1 P_2} x_1 x_2 \, dx_1 dx_2$$

$$= \frac{1}{24} J (2 x_{10} x_{20} + 2 x_{11} x_{21} + 2 x_{12} x_{22}$$

$$+ x_{10} x_{21} + x_{10} x_{22} + x_{11} x_{20} + x_{11} x_{22} + x_{12} x_{20} + x_{12} x_{21})$$

$$= J \frac{1}{24}$$

$$\begin{array}{ccc} (2 x_{10} x_{20} & +x_{10} x_{21} & +x_{10} x_{22} \\ +x_{11} x_{20} & +2 x_{11} x_{21} & +x_{11} x_{22} \\ +x_{12} x_{20} & +x_{12} x_{21} & +2 x_{12} x_{22}) \end{array} \qquad (7.17)$$

7.4 Simplex Integration for Two Dimensional Manifold Method

Since simplex integrations always have the Jacobian J as factor and J is an oriented area, the integrations on positive area and negative area can be neutralized. Denote

$$P_1 P_2 P_3 \cdots P_n \qquad P_{n+1} = P_1$$

with

$$P_i = (x_{1i}, x_{2i})$$

as a polygon, the integrations (7.14)-(7.17) on

$$P_1 P_2 P_3 \cdots P_n$$

are computed by summations. (7.18)-(7.21) are the integrations of manifold method. Here P_0 can be any point, $P_0 = (0,0)$ are chosen in order to have simpler formulae. Computed by simplex integrations, integrals (7.18)-(7.21) are represented by the coordinates of the boundary vertices only.

$$\int\int_{(A)} dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(0,0)$$

$$= \frac{1}{2} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix} \tag{7.18}$$

$$\int\int_{(A)} x_1 dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(1,0)$$

$$\int\int_{(A)} x_1 dx_1 dx_2 = \frac{1}{6} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix} (x_{1\,k} + x_{1\,k+1})$$

$$\int\int_{(A)} x_2 dx_1 dx_2 = \frac{1}{6} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix} (x_{2\,k} + x_{2\,k+1}) \tag{7.19}$$

$$\int\int_{(A)} x_1^2 dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(2,0)$$

$$= \frac{1}{12} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(x_{1\,k}^2 + x_{1\,k+1}^2 + x_{1\,k} x_{1\,k+1})$$

$$\int\int_{(A)} x_1^2 dx_1 dx_2 = \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$\begin{pmatrix} 2x_{1\,k} x_{1\,k} & +x_{1\,k} x_{1\,k+1} \\ +x_{1\,k+1} x_{1\,k} & +2x_{1\,k+1} x_{1\,k+1} \end{pmatrix}$$

$$\int\int_{(A)} x_2^2 dx_1 dx_2 = \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$\begin{pmatrix} 2x_{2\,k} x_{2\,k} & +x_{2\,k} x_{2\,k+1} \\ +x_{2\,k+1} x_{2\,k} & +2x_{2\,k+1} x_{2\,k+1} \end{pmatrix} \tag{7.20}$$

$$\int\int_{(A)} x_1 x_2 dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(1,1)$$

$$= \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(2x_{1\,k} x_{2\,k} + 2x_{1\,k+1} x_{2\,k+1} + x_{1\,k} x_{2\,k+1} + x_{1\,k+1} x_{2\,k})$$

$$\int\int_{(A)} x_1 x_2 dx_1 dx_2 = \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$\begin{pmatrix} 2x_{1\,k} x_{2\,k} & +x_{1\,k} x_{2\,k+1} \\ +x_{1\,k+1} x_{2\,k} & +2x_{1\,k+1} x_{2\,k+1} \end{pmatrix} \tag{7.21}$$

Chapter 8

Equation Solver and Open-Close Iterations for Inertia Dominant Equilibrium Equations

8.1 Time Step Based Large Displacement Analysis

The manifold method computations follow the time steps. The time steps are small enough that the second order displacements are neglected. All the geometric and physical parameters have to be transferred from the end of the previous time step to the beginning of the next time step. The following items are to be transferred:

[1] stresses of each element,

[2] strains of each element,

[3] velocities of each element,

[4] geometry of the joint boundaries and elements,

[5] all closed contacts

The position and state parameters of the closed contacts have to go to the next step. The geometric parameters and physical parameters of contacts will be transferred:

[1] the contact vertex and edge,

[2] the position of contact point,

[3] the normal displacement and normal force,

[4] the shear displacement and shear force,

[5] locking or sliding as contact state.

The manifold method computes both statics and dynamics by time steps. For large displacements and deformations, the statics is the ultimate stabilized state of dynamics af-

ter sufficient long times. Such a stabilized state can be reach only if some kind of energy reduction is counted in the computation.

The previously customized one-step static computation is correct only if the second order displacements can be omitted. In such a case, physically one infinite large time step Δ is chosen under the small ultimate displacements. The inertia force would be proportional with

$$\frac{u}{\Delta^2} \tag{8.1}$$

The value of (8.1) is zero if time step Δ is very large, then the inertia force is zero. This is the reason why the one step statics do not consider the inertia force or mass matrices.

The current version of the manifold code is a simple version. In the beginning of a time step, the dynamic computations inherit the velocity of the end of the last step. For static computation it is assumed that, the initial velocity of each time step is zero. The current static computation is obviously incomplete, the way of energy reduction has to be studied in the future.

8.2 Open-Close Iterations

Within each time step, the global equations have to be solved repeatedly while selecting the lock positions. The procedure of adding and removing stiff springs is open-close iteration.

If a contact has a tensile contact force from the normal spring, the two sides will separate after the removal of this stiff spring. If the vertex penetrated the edge in other side of the contact, a stiff spring is applied.

For each contact, there are three modes: open, sliding and lock. The criteria of the mode changing is in the following.

mode change	condition
open - open	$N > 0$
open - slide	$N < 0$ and $|T| > tan\phi|N|$
open - lock	$N < 0$ and $|T| < tan\phi|N|$
slide - open	$N > 0$

slide - slide	$N < 0$ and $\vec{t} \parallel -\vec{f}$
slide - lock	$N < 0$ and $\vec{t} \parallel \vec{f}$
lock - open	$N > 0$
lock - slide	$N < 0$ and $|T| > tan\phi|N|$
lock - lock	$N < 0$ and $|T| < tan\phi|N|$

where p is spring stiffness

$\vec{n}$ is normal displacement vector pointing vertex

N is normal displacement, $N > 0$ is open

$\vec{t}$ is shear displacement vector pointing $\overrightarrow{P_2 P_3}$

T is shear displacement, $T > 0$ if $\vec{t}$ in same direction as $\overrightarrow{P_2 P_3}$

$\parallel$ means two vectors point the same direction

$\vec{t}$ is friction force vector pointing $\overrightarrow{P_2 P_3}$

ϕ friction angle

Finding the mode changing, the following operations will be done:

mode change	operation
open - open	no changing
open - slide	apply pair of friction forces
open - lock	apply normal and shear springs
slide - open	delete friction forces
slide - slide	no changing
slide - lock	delete friction forces and apply shear spring
lock - open	delete normal and shear springs
lock - slide	delete shear spring and apply friction forces
lock - lock	no changing

The open-close iterations to ensue

[1] no-penetrations in the open contacts,

[2] no-tensions in the contacts with normal springs.

The two conditions have to be fulfilled in all contacts. If the two conditions are not fulfilled after five times of open-close iterations, the time step will be reduced to one third, and the open-close iteration continues.

8.3 SOR Iteration Method

The abbreviation "SOR" stands for successive over relaxation method. This method is for solving linear equations,

$$[A][X] = [F] \tag{8.2}$$

Equation (8.2) has submatrix structure

$$
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1n} \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2n} \\
A_{31} & A_{32} & A_{33} & A_{34} & \cdots & A_{3n} \\
A_{41} & A_{42} & A_{43} & A_{44} & \cdots & A_{4n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & A_{n3} & A_{n4} & \cdots & A_{nn}
\end{pmatrix}
\begin{pmatrix}
X_1 \\ X_2 \\ X_3 \\ X_4 \\ \vdots \\ X_n
\end{pmatrix}
=
\begin{pmatrix}
F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_n
\end{pmatrix}. \tag{8.3}
$$

where

[1] [A] is a $n \times n$ coefficient matrix,

[2] [X] is a $n \times 1$ unknown matrix ,

[3] [F] is a $n \times 1$ matrix of free terms. The elements of these matrices are still submatrices.

[1] The elements A_{ij} of matrix [A] are $q \times q$ matrices,

[2] The matrix [A] is symmetric, $[A_{ij}]^T = [A_{ji}]$

[3] The elements X_i of matrix [X] are $q \times 1$ matrices,

[4] The elements F_i of matrix [F] are $q \times 1$ matrices.

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Denote the diagonal matrix $[D]$ as

$$[D] = \begin{pmatrix} A_{11} & 0 & 0 & 0 & \dots & 0 \\ 0 & A_{22} & 0 & 0 & \dots & 0 \\ 0 & 0 & A_{33} & 0 & \dots & 0 \\ 0 & 0 & 0 & A_{44} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & A_{nn} \end{pmatrix}$$

Then there are

$$(([A] - [D]) + [D])[X] = [F]$$
$$[D][X] = [F] - ([A] - [D])[X]$$
$$[X] = [D]^{-1}[F] - [D]^{-1}([A] - [D])[X]$$
$$[X] = [G] - [B][X] \qquad (8.4)$$

where

$$[G] = [D]^{-1}[F])$$
$$[B] = [D]^{-1}([A] - [D])$$

[1] [B] is a $n \times n$ coefficient matrix,

[2] [G] is a $n \times 1$ force matrix ,

[3] [X] is a $n \times 1$ unknown matrix , The elements of these matrices are still submatrices.

[1] The elements D_{ii} of matrix $[D]$ are $q \times q$ matrices,

[2] The elements B_{ij} of matrix $[B]$ are $q \times q$ matrices,

[3] The elements G_i of matrix $[G]$ are $q \times 1$ matrices.

$$\begin{cases} [D_{ii}] = [A_{ii}], & \\ [D_{ij}] = [0], & if\, i \neq j \\ [G_{ij}] = [D_{ii}]^{-1}[F_{ij}], & \\ [B_{ij}] = [D_{ii}]^{-1}[A_{ij}], & if\, i \neq j \\ [B_{ii}] = [0]. & \end{cases}$$

For SOR method the factor of over relaxation is ω,

$$1 < \omega < 2$$

Denote $[X_i]^{(m)}$ is the solution after m iterations, then the solution $[X_i]^{(m+1)}$ of the next iteration is

$$[X_i]^{(m+1)} = [X_i]^{(m)} + \omega([G_i] - \sum_{k=1}^{i-1}[B_{ik}][X_k]^{(m+1)} - \sum_{k=i+1}^{n}[B_{ik}][X_k]^{(m)} - [X_i]^{(m)})$$

$$[X_i]^{(m+1)} = (1 - \omega)[X_i]_i^{(m)} + \omega([G_i] - \sum_{k=1}^{i-1}[B_{ik}][X_k]^{(m+1)} - \sum_{k=i+1}^{n}[B_{ik}][X_k]^{(m)})$$

$$\tag{8.5}$$

The matrix form of the SOR method is

$$
\begin{pmatrix} X_1^{(m+1)} \\ X_2^{(m+1)} \\ X_3^{(m+1)} \\ \vdots \\ X_n^{(m+1)} \end{pmatrix}
= (1-\omega) \begin{pmatrix} X_1^{(m)} \\ X_2^{(m)} \\ X_3^{(m)} \\ \vdots \\ X_n^{(m)} \end{pmatrix}
+ \omega \begin{pmatrix} G_1 \\ G_2 \\ G_3 \\ \vdots \\ G_n \end{pmatrix}
$$

$$
-\omega \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ B_{21} & 0 & 0 & 0 & \cdots & 0 \\ B_{31} & B_{32} & 0 & 0 & \cdots & 0 \\ B_{41} & B_{42} & B_{43} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & B_{n3} & B_{n4} & \cdots & 0 \end{pmatrix}
\begin{pmatrix} X_1^{(m+1)} \\ X_2^{(m+1)} \\ X_3^{(m+1)} \\ \vdots \\ X_n^{(m+1)} \end{pmatrix}
$$

$$
-\omega \begin{pmatrix} 0 & B_{12} & B_{13} & B_{14} & \cdots & B_{1n} \\ 0 & 0 & B_{23} & B_{24} & \cdots & B_{2n} \\ 0 & 0 & 0 & B_{34} & \cdots & B_{3n} \\ 0 & 0 & 0 & 0 & \cdots & B_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}
\begin{pmatrix} X_1^{(m)} \\ X_2^{(m)} \\ X_3^{(m)} \\ \vdots \\ X_n^{(m)} \end{pmatrix}
\tag{8.6}
$$

8.4 Simple Iteration Method

The simple iteration method of equation

$$[A][X] = [F]$$

has a simpler algorithm. Generally speaking, the simple iteration method is mush more slower than SOR iteration method. The formula of simple iteration method is mathematically clear and clean. However, practically this method never became a major equation solver, because of its low efficiency comparing with the other iteration methods. It seems this method will stay in textbook forever to give the beginners a fresh idea about what is iteration method.

It is often, that things can change in an unexpected manner. Two thing happened recently:

[1] parallel computation,

[2] inertia dominant computation in discontinuous deformation analysis.

Now, simple iteration method is the equation solver which is most suitable to the parallel computation. In the mean time, the diagonal dominant matrices of DDA and manifold method are good enough to use the simple iteration method. Choosing small time steps, the coefficient matrices can be almost diagonal matrices, then, there will be almost no difference between simple iteration method and SOR iteration method.

As before, the equation is

$$
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & A_{14} & \cdots & A_{1n} \\
A_{21} & A_{22} & A_{23} & A_{24} & \cdots & A_{2n} \\
A_{31} & A_{32} & A_{33} & A_{34} & \cdots & A_{3n} \\
A_{41} & A_{42} & A_{43} & A_{44} & \cdots & A_{4n} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & A_{n3} & A_{n4} & \cdots & A_{nn}
\end{pmatrix}
\begin{pmatrix}
X_1 \\ X_2 \\ X_3 \\ X_4 \\ \vdots \\ X_n
\end{pmatrix}
=
\begin{pmatrix}
F_1 \\ F_2 \\ F_3 \\ F_4 \\ \vdots \\ F_n
\end{pmatrix}
.
$$

The algorithm of the simple iteration method is

$$
\begin{pmatrix}
X_1^{(m+1)} \\
X_2^{(m+1)} \\
X_3^{(m+1)} \\
\vdots \\
X_n^{(m+1)}
\end{pmatrix}
$$

$$= - \begin{pmatrix} A_{11}^{-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_{33}^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_{44}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{nn}^{-1} \end{pmatrix}$$

$$\begin{pmatrix} 0 & A_{12} & A_{13} & A_{14} & \cdots & A_{1n} \\ A_{21} & 0 & A_{23} & A_{24} & \cdots & A_{2n} \\ A_{31} & A_{32} & 0 & A_{34} & \cdots & A_{3n} \\ A_{41} & A_{42} & A_{43} & 0 & \cdots & A_{4n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & A_{n3} & A_{n4} & \cdots & 0 \end{pmatrix} \begin{pmatrix} X_1^{(m)} \\ X_2^{(m)} \\ X_3^{(m)} \\ \vdots \\ X_n^{(m)} \end{pmatrix}$$

$$+ \begin{pmatrix} A_{11}^{-1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & A_{33}^{-1} & 0 & \cdots & 0 \\ 0 & 0 & 0 & A_{44}^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{nn}^{-1} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{pmatrix} \qquad (8.7)$$

8.5 Time Step Algorithm and Iteration Convergence

The principles of choosing time steps Δ are

[1] Δ is small enough, so that the second order displacements are neglected;

[2] Δ is small enough, so that the SOR iteration will converge in less than 30 iterations;

[3] Δ is small enough, so that the open-close iterations will converge in less than 6 iterations,

[4] Δ is large enough, so that the computation will represent larger time span and the displacements are stabilized if possible.

There are three options to define time step:

[1] directly enter the time step,

[2] enter the allowable maximum step displacement, then define time step from the maximum allowable step displacement,

[3] compute the maximum allowable step displacement, then define time step from the maximum allowable step displacement.

Denote

$\mathcal{D}$ as the maximum allowed step displacement,

$\mathcal{U}$ as the mass of the average elements,

$\mathcal{F}$ as the maximum load on the average elements,

$\mathcal{V}$ as the maximum velocity of elements,

The equation of time step Δ is

$$\mathcal{G}\Delta^2 + \mathcal{V}\Delta - \mathcal{D} = 0$$

let

$$\mathcal{G} = \frac{\mathcal{F}}{\mathcal{U}}$$

$$\Delta = \frac{1}{2\mathcal{G}}(-\mathcal{V} + \sqrt{\mathcal{V}^2 + 4\mathcal{G}\mathcal{D}})$$

Chapter 9

Applications of Numerical Manifold Method

9.1 Joint Computations

Figure 9.1 shows the ability of the manifold method to compute a joint or fracture.

Figure 9.2 shows the deformations of a domain with a self intersected curved joint

9.2 Block Computations

Figure 9.3 shows the failure of an arch under the point load on the center block and self weight.

Figure 9.4 shows the failure of a gravity dam with rock foundation. The loads are the upstream water pressure and the self weight of the dam.

9.3 Slope Sliding

Figure 9.5 is the result of slope sliding of rock blocks. It can be noticed that, the center block separated two adjacent blocks during the sliding. The result is consistent with the lab test.

Figure 9.6 is a soil slope which slides along a circular surface. The circle sliding computation satisfies all equilibrium conditions.

9.4 Failure of Structures

Figure 9.7 shows the deformation of the joints, blocks and the continuous materials of double simply supported beams

Figure 9.8 shows the deformation of the joints, blocks and the continuous materials of jointed double cantilever beams
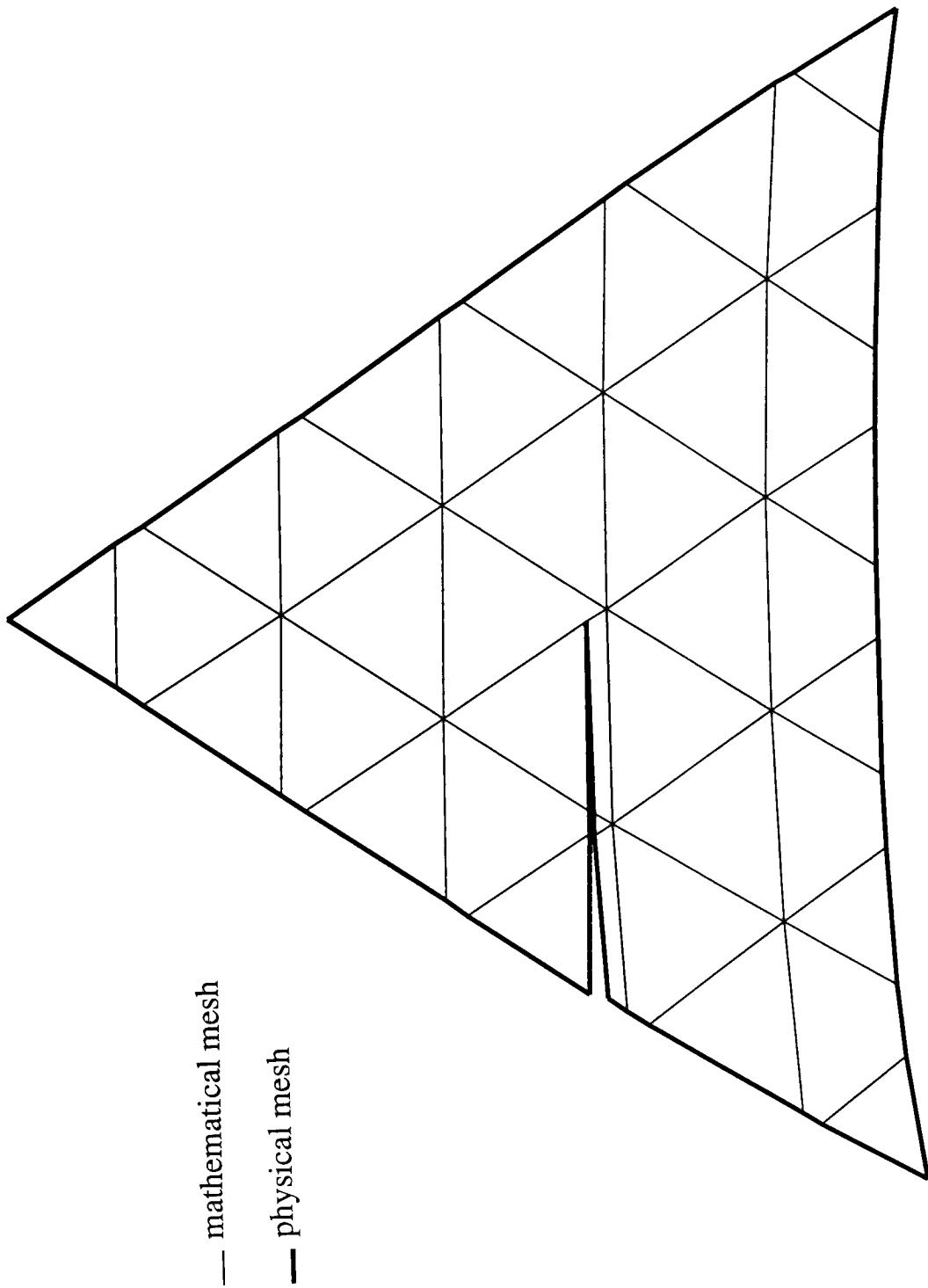
FIGURE 9.1 Single joint in a triangular shape domain

— mathematical mesh

— physical mesh
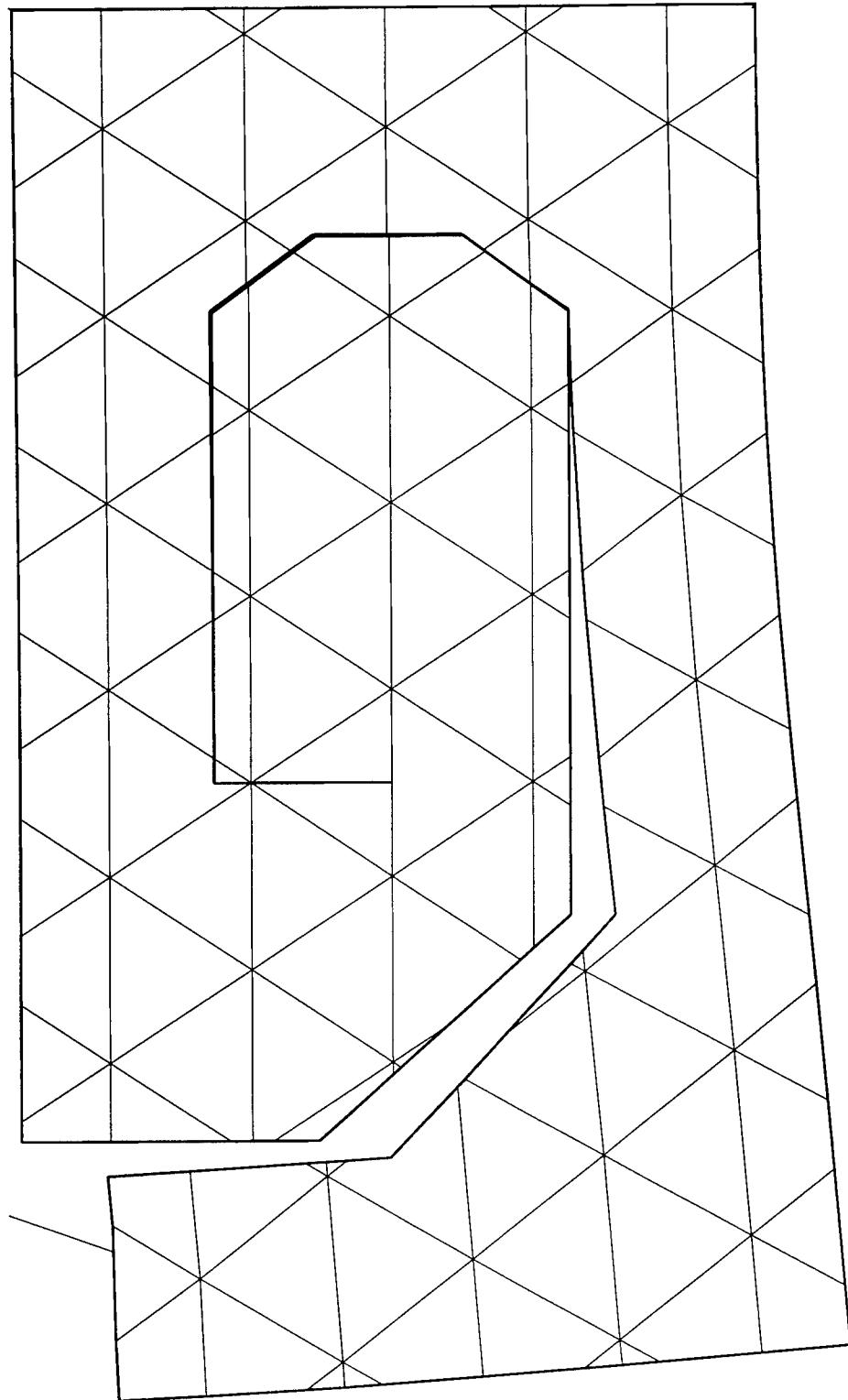
FIGURE 9.2 Deformations of a domain with a complex joint
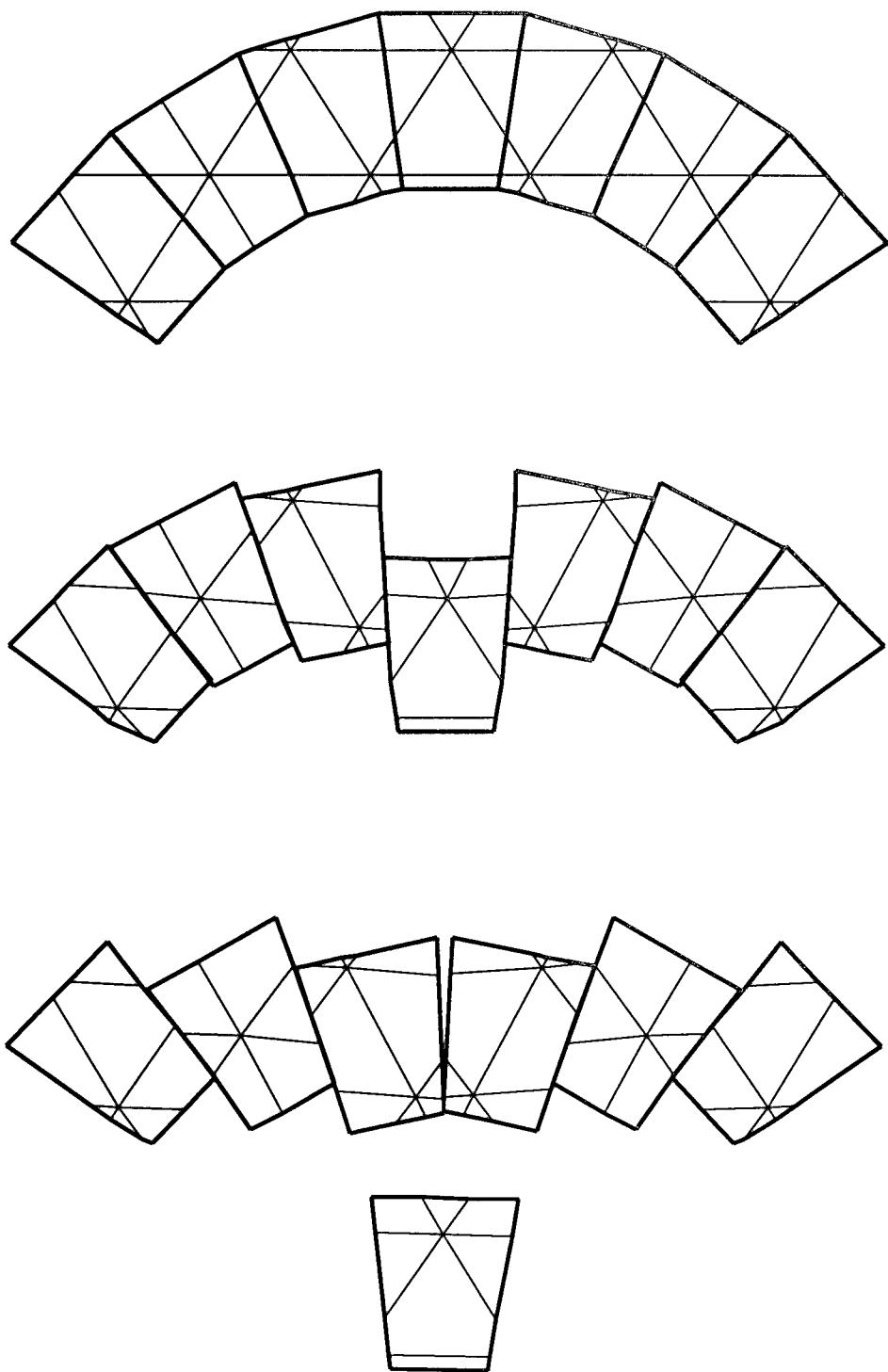
FIGURE 9.3 The failure of a deformable arch
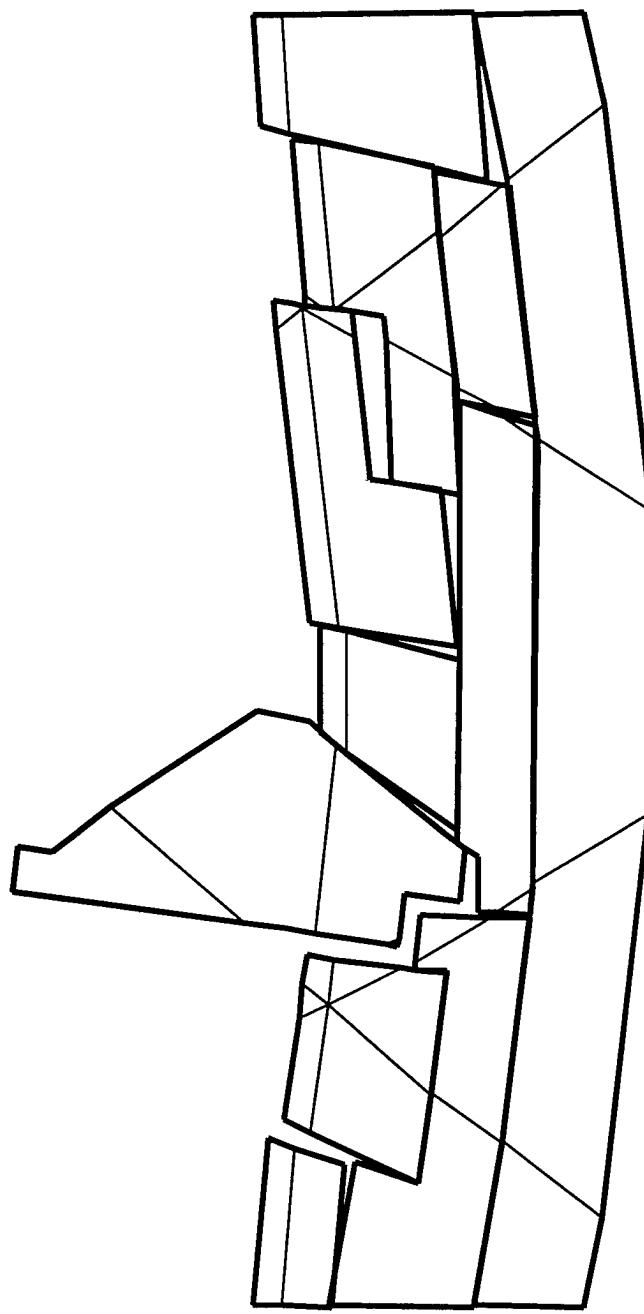
FIGURE 9.4 Failure of dam and rock foundations
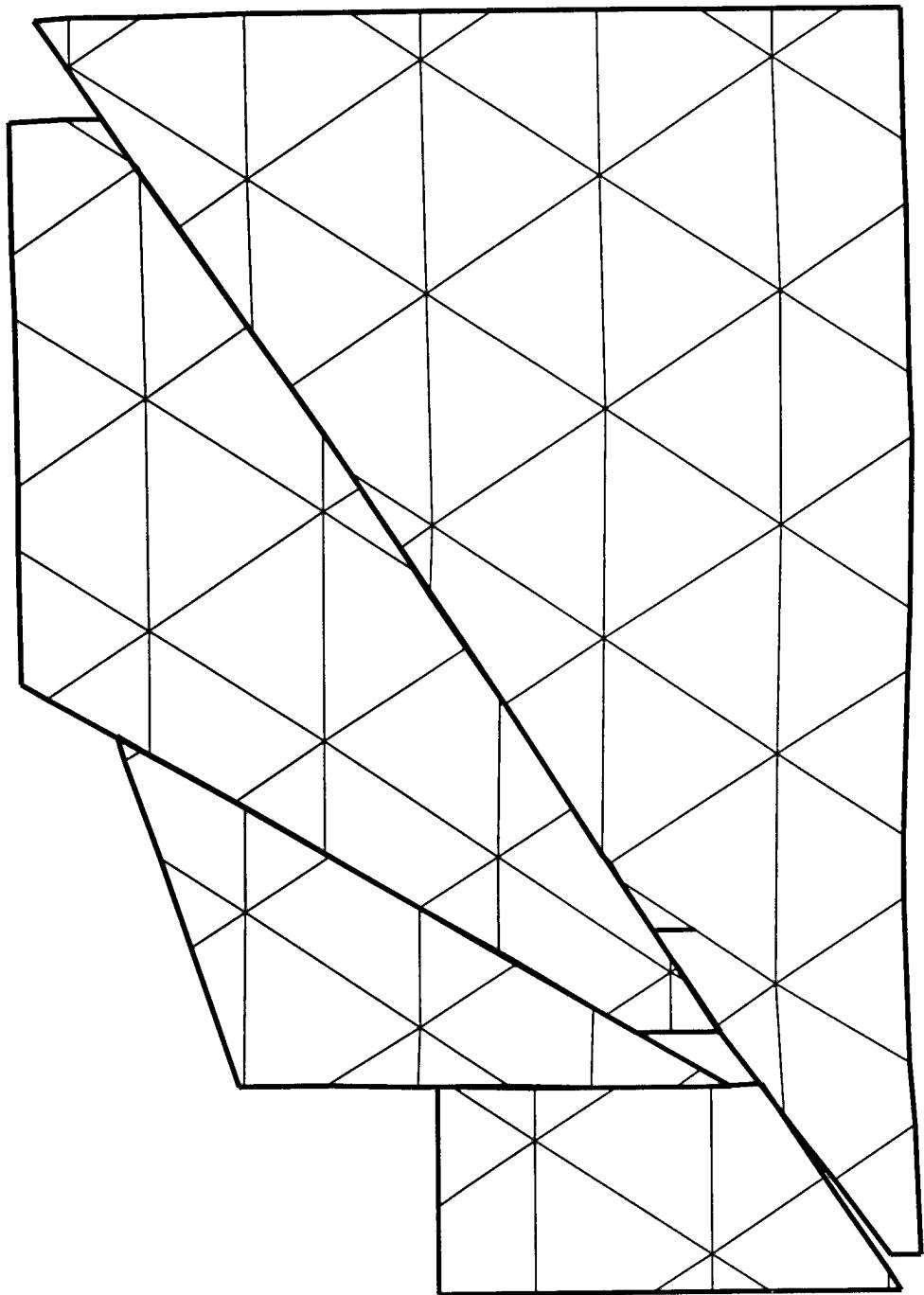
FIGURE 9.5 Multi-block sliding
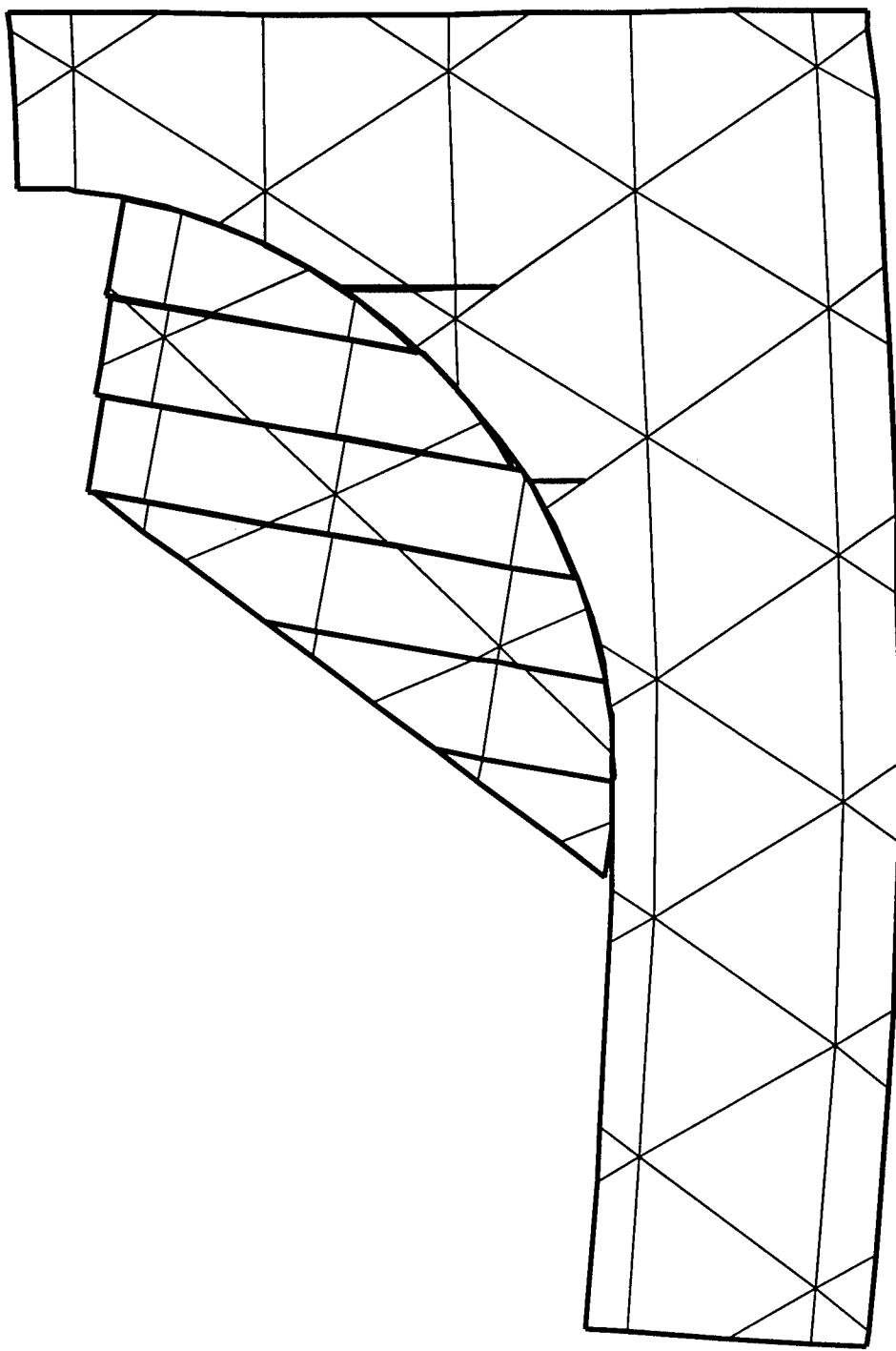
FIGURE 9.6 Circular sliding
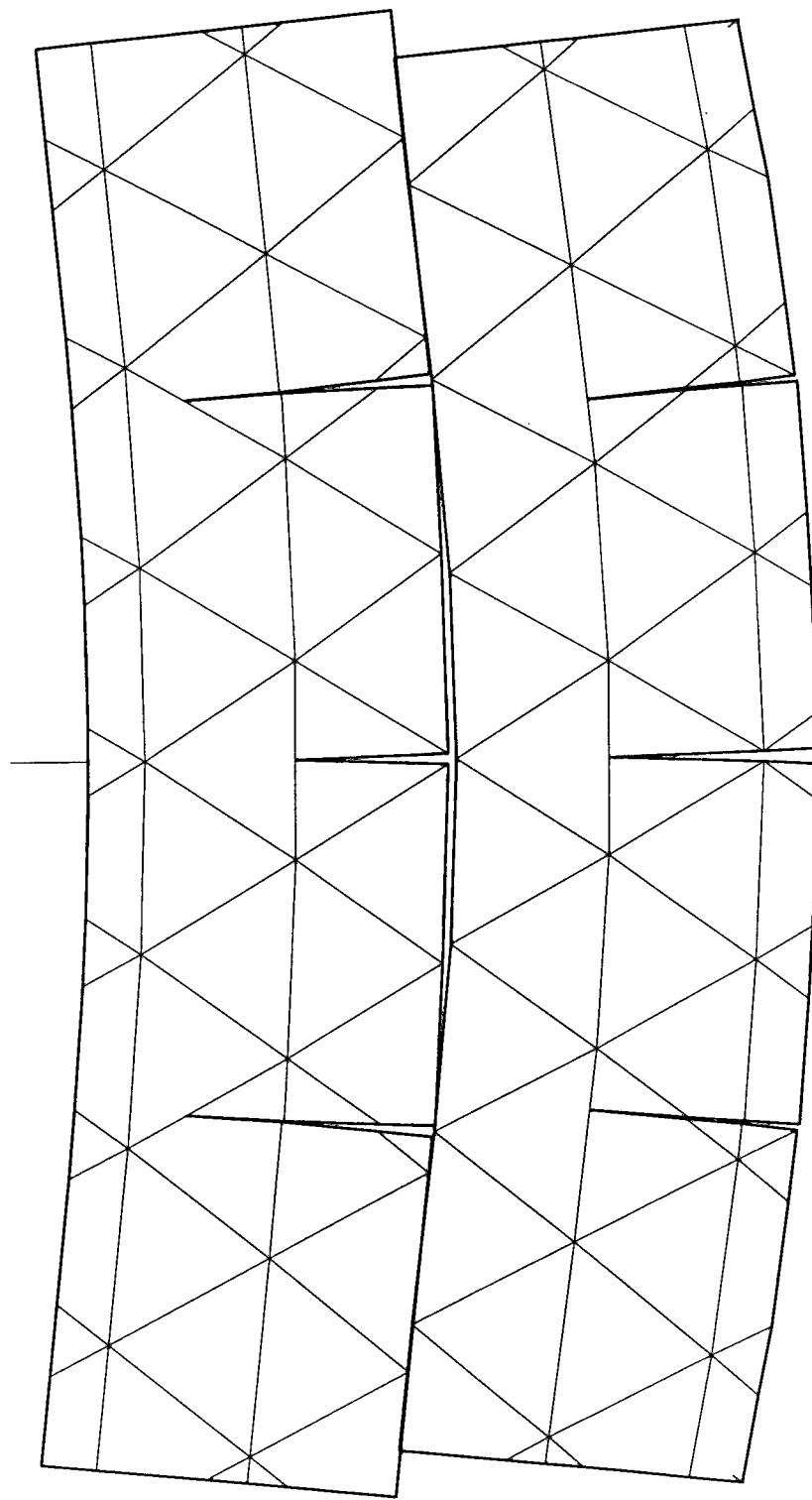
FIGURE 9.7 Deformation of double simply supported beams
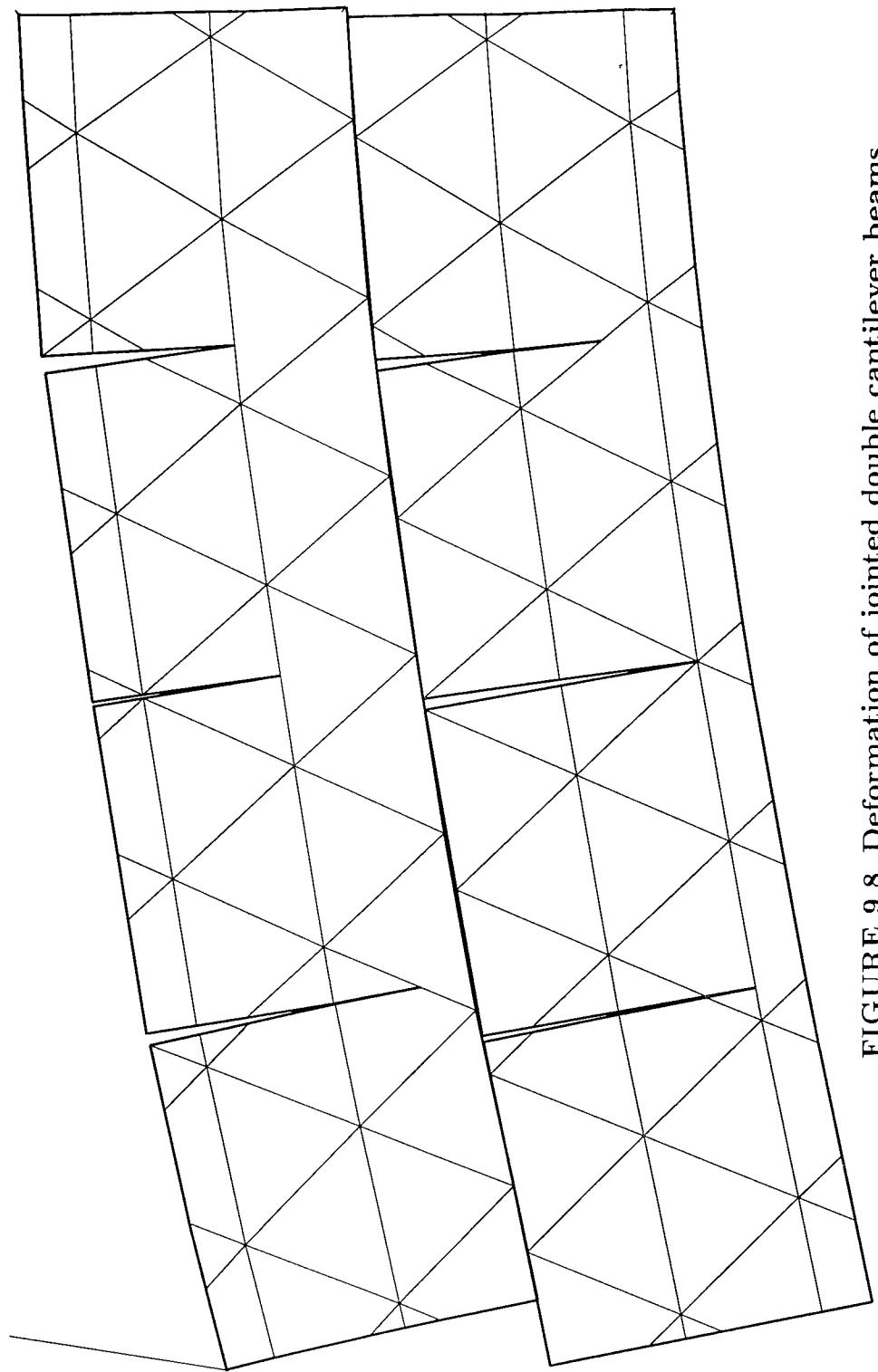
FIGURE 9.8 Deformation of jointed double cantilever beams

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9.5 Conclusions

This new theory, entitled the Manifold Method of Material Analysis, combines physical meshes and mathematical meshes. These physical meshes provide the means to consider both jointed and continuous materials, and even different material phases (i. e. solid, gas or liquid). At present, a theory for the manifold method has been accomplished, as has a first generation 2-D dynamic computer code. The preliminary results are encouraging (for example, the convergence of the solutions has been established). Finite element and DDA formulations are special cases of this developing theory. A brief listing of a few of the advantages of the manifold method follows:

[1] free surface and flexible boundaries

[2] analysis not hindered by boundary conditions

[3] free form elements (any shape)

[4] conservation of energy

[5] obeys Coulomb's law

[6] very small to very large deformation

[7] statics and dynamics possible

[8] analytically correct

[9] continuous and discontinuous analysis

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Appendix: The mathematical manifold

The " Manifold " is a main subject of differential geometry and topology in modern mathematics. The basic structure of the manifolds is a finite cover system and the connection functions among the covers. Different from the mesh, the covers can be overlapped or folded to present a combinatorial space. Therefore the manifold method is to pursue global solutions on global spaces. There were no direct connections before between manifolds and engineering analysis. The essential difficulty is the complicated boundaries and discontinuous interfaces of the real engineering cases.

Under the generalized definition of manifold, there are two independent mesh systems now: mathematical mesh and physical mesh. The mathematical mesh consists of the folded covers. Independent cover functions are defined on each cover. The weight functions connect all locally defined functions to a combinatorial global function.

Acknowledgment

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Simplex Integration for Manifold Method and Discontinuous Deformation Analysis

Gen-hua Shi

Geotechnical Lab, US Army Engineer Waterways Experiment Station

Vicksburg, MS 39180-6199

Abstract

At least every engineer has to compute the volume of generally shaped blocks. Is there a formula where the volume is precisely represented by the coordinates of boundary vertices? If block movements are considered, the center of gravity has to be computed? Is there a formula where the center of gravity is also represented by the coordinates of boundary vertices? The simplex integration developed for DDA computation can also solve these questions. The convergency and accuracy of DDA algorithms depend upon mainly the analytical integrations on complex shapes. Simplex integrations are accurate solution on n-dimensional generally shaped domains. The integrand could be any n-dimensional polynomials. The DDA computations of three rock failure cases are presented.

Simplex Integration on a Simplex

Simplex has the most simple shape in $1, 2, 3, \ldots, n$ dimensional space. Different from the ordinary integration, simplex integration has only simplex as integral domain. Simplex also has positive or negative orientations. Positive or negative orientations define positive or negative volumes respectively.
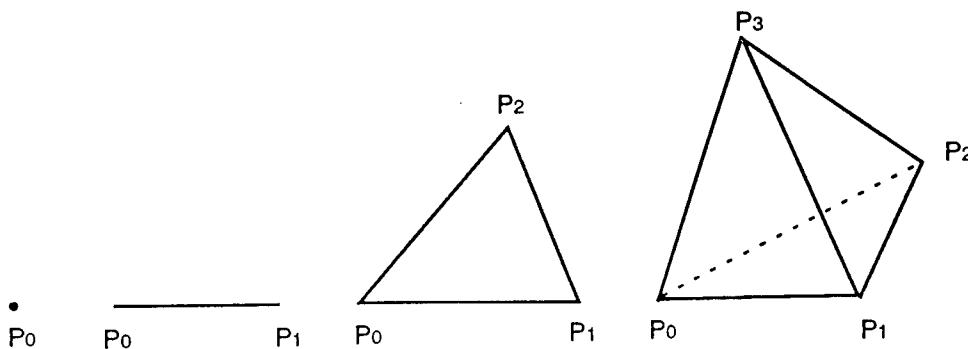

Figure 1. 0,1,2,3 dimensional implex

The 0 dimensional simplex is a point P_0, its volume is

$$\frac{1}{0!} \mid 1 \mid = 1$$

The 0-d simplex integration can be considered as a normal real number.

One dimensional simplex $P_0 P_1$ is an oriented segment, its volume is

$$\frac{1}{1!} \begin{vmatrix} 1 & x_{10} \\ 1 & x_{11} \end{vmatrix} = x_{11} - x_{10}$$

The volume of the simplex $P_1 P_0$ is the negative volume of the simplex $P_0 P_1$. Ordinary 1-d integrations are simplex integrations:

$$\int_{P_0}^{P_1} f(x)dx = - \int_{P_1}^{P_0} f(x)dx$$

Therefore, for any co-line point P_2,

$$\int_{P_0}^{P_2} f(x)dx + \int_{P_2}^{P_1} f(x)dx = \int_{P_0}^{P_1} f(x)dx + \int_{P_1}^{P_2} f(x)dx + \int_{P_2}^{P_1} f(x)dx = \int_{P_0}^{P_1} f(x)dx$$

The 1-d integration addition is the same as vector addition. The integrations on negative vectors and positive vectors can be nullified.

Two dimensional simplex $P_0 P_1 P_2$ is a oriented triangle, its volume is

$$\frac{1}{2!} \begin{vmatrix} 1 & x_{10} & x_{20} \\ 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \end{vmatrix}$$

The volume of simplex $P_1 P_0 P_2$ is the negative volume of simplex $P_0 P_1 P_2$.

Three dimensional simplex $P_0 P_1 P_2 P_3$ is a oriented tetrahedron, its volume is

$$\frac{1}{3!} \begin{vmatrix} 1 & x_{10} & x_{20} & x_{30} \\ 1 & x_{11} & x_{21} & x_{31} \\ 1 & x_{12} & x_{22} & x_{32} \\ 1 & x_{13} & x_{23} & x_{33} \end{vmatrix}$$

The volume of simplex $P_1 P_0 P_2 P_3$ is the negative volume of simplex $P_0 P_1 P_2 P_3$.

Unfortunately, two dimensional and three dimensional ordinary integrations are volume integration, where the volume is always positive. For the 2 or 3 dimensional ordinary

integration, the integration domain has no orientation, therefore the integration has no algebraic addition of oriented domain. It is then necessary to introduce simplex integration on a simplex, which is a simple oriented domain.

The simplex integration is not intended to do integrations on simplex only. Obviously a complex shape always can be subdivided into simplex, so simplex integration can be computed in each simplex, the summation of simplex integrations is the ordinary integration over the complex shape. However this is also not the way of using simplex integration.

Giving a polygon $P_1 P_2 P_2 P_4 P_5 P_6$ with $P_6 = P_1$, such that P_i rotate at the same direction as from ox to oy. For any point P_0, the algebraic addition of the 2-d simplex volume (area) $P_0 P_1 P_2$, $P_0 P_2 P_3$, $P_0 P_3 P_4$, $P_0 P_4 P_5$ and $P_0 P_5 P_1$ is the area A of the polygon. Let $P_0 = (0,0)$,

$$A = \frac{1}{2} \sum_{i=1}^{5} \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_{1\,i} & x_{2\,i} \\ 1 & x_{1\,i+1} & x_{2\,i+1} \end{vmatrix} = \frac{1}{2} \sum_{i=1}^{5} \begin{vmatrix} x_{1\,i} & x_{2\,i} \\ x_{1\,i+1} & x_{2\,i+1} \end{vmatrix}$$

In Figure 2, the area of simplex $P_0 P_2 P_3$, $P_0 P_3 P_4$, $P_0 P_4 P_5$ and $P_0 P_5 P_1$ are positive; the area of simplex $P_0 P_1 P_2$ is negative. The algebraic sum is exactly the area of polygon $P_1 P_2 P_2 P_4 P_5 P_6$. The area A is then represented by the coordinates of boundary vertices.
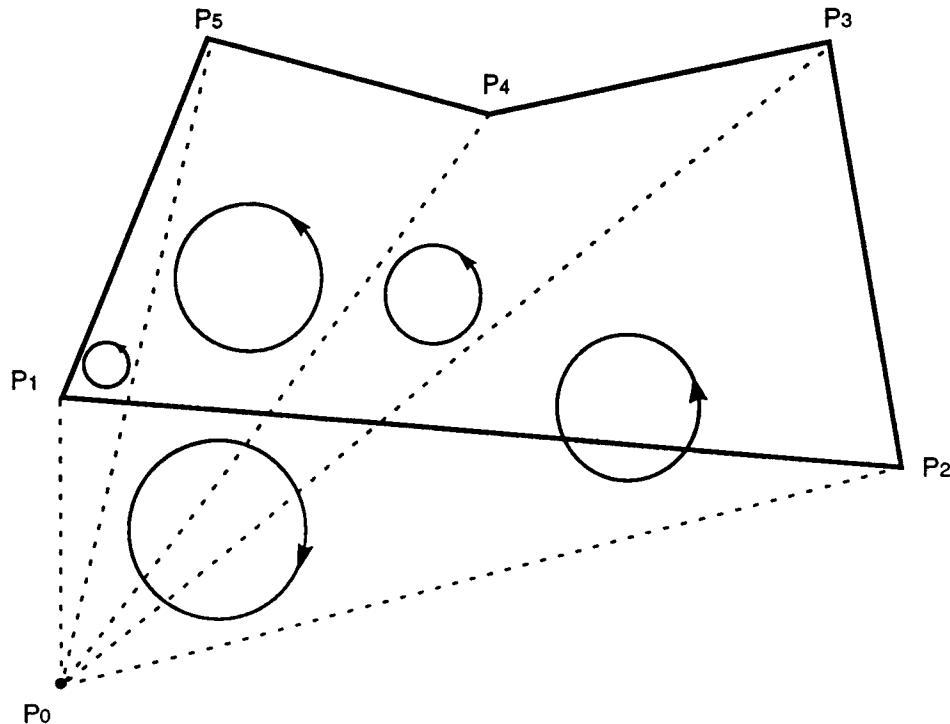

Figure 2. Addition of plus or minus area of simplex   131

In general, simplex integration can compute ordinary integrations without subdividing 2-d domains to triangles and 3-d volumes to tetrahedrons. FEM mesh is unnecessary for simplex integration. Using simplex integration, the integration of any n-dimensional polynomials can be represented by the coordinates of boundary vertices of generally shaped polyhedron.

Two Dimensional Simplex Integrations

The two dimensional integrations here in this section are used for two dimensional DDA algorithm. Since the displacement function is linear function of coordinates (x, y), the integrands are of degree 0, 1, 2.

A 2 dimensional simplex has 3 vertices

$$P_0, P_1, P_2.$$

$$
\begin{aligned}
P_0 &: \quad (x_{10}, \quad x_{20}) \\
P_1 &: \quad (x_{11}, \quad x_{21}) \\
P_2 &: \quad (x_{12}, \quad x_{22})
\end{aligned}
$$

The 2 dimensional coordinate simplex has 3 vertices

$$U_0, U_1, U_2.$$

$$
\begin{aligned}
U_0 &: \quad (\ 0, \quad 0) \\
U_1 &: \quad (\ 1, \quad 0) \\
U_2 &: \quad (\ 0, \quad 1)
\end{aligned}
$$

The following coordinate transformation

$$(u_1, u_2) \rightarrow (x_1, x_2)$$

s coordinate simplex

$$U_0 U_1 U_2$$

to normal simplex

$$P_0 P_1 P_2$$

$$
\begin{aligned}
x_1 &= x_{10} \left(1 - \sum_1^2 u_i\right) + x_{11} \, u_1 + x_{12} \, u_2 \\
x_2 &= x_{20} \left(1 - \sum_1^2 u_i\right) + x_{21} \, u_1 + x_{22} \, u_2
\end{aligned}
$$

The Jacobi determinant is

$$J = \frac{D(x_1,\, x_2)}{D(u_1,\, u_2)}$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix}$$

$$= \begin{vmatrix} x_{11} - x_{10} & x_{12} - x_{10} \\ x_{21} - x_{20} & x_{22} - x_{20} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_{10} & x_{20} \\ 1 & x_{11} & x_{21} \\ 1 & x_{12} & x_{22} \end{vmatrix}$$

Since $P_0 P_1 P_2$ is a 2-dimensional simplex with non-zero volume, J is non-zero. Translation can be rewritten as

$$(u_0, u_1, u_2) \rightarrow (1, x_1, x_2)$$

$$
\begin{array}{ccccccc}
1 & = & u_0 & + & u_1 & + & u_2 \\
x_1 & = & x_{10}\, u_0 & + & x_{11}\, u_1 & + & x_{12}\, u_2 \\
x_2 & = & x_{20}\, u_0 & + & x_{21}\, u_1 & + & x_{22}\, u_2
\end{array}
$$

Two dimensional simplex integration

$$S_{P_0 P_1 P_2}(m_1, m_2)$$

on a two dimensional simplex

$$P_0 P_1 P_2$$

is defined as normal integration times the sign of determinant J.

$$
\begin{aligned}
&S_{P_0 P_1 P_2}(m_1, m_2) \\
&= sign(J) \int\!\!\int_{P_0 P_1 P_2} x_1^{m_1} x_2^{m_2}\, dx_1 dx_2 \\
&= sign(J) \int\!\!\int_{U_0 U_1 U_2} x_1^{m_1} x_2^{m_2}\, |J| du_1 du_2 \\
&= J \int\!\!\int_{U_0 U_1 U_2} x_1^{m_1} x_2^{m_2}\, du_1 du_2
\end{aligned}
$$

Here

$$x_l^{m_l} = \left(\sum_{k=0}^{2} x_{lk} u_k\right)^{m_l}$$

then the two dimensional simplex integration can be represented by the following basic forms of simplex integration:

$$S_1 = \int\int_{U_0 U_1 U_2} u_0^{i_0} u_1^{i_1} u_2^{i_2} du_1 du_2$$

$$= \int\int_{u_0, u_1, u_2 \geq 0}^{u_0 + u_1 + u_2 = 1} u_0^{i_0} u_1^{i_1} u_2^{i_2} du_1 du_2$$

$$= \int_0^1 \int_0^{1-u_2} u_2^{i_2} u_1^{i_1} u_0^{i_0} du_1 du_2$$

$$= \int_0^1 u_2^{i_2} \left(\int_0^{1-u_2} u_1^{i_1} (1 - u_2 - u_1)^{i_0} du_1 \right) du_2$$

After i_1 times of integration by parts, the inner integration can be computed.

$$\int_0^{1-u_2} u_1^{i_1} (1 - u_2 - u_1)^{i_0} du_1$$

$$= \int_0^{1-u_2} d(\frac{1}{i_1 + 1} u_1^{i_1+1})(1 - u_2 - u_1)^{i_0}$$

$$= \frac{1}{i_1 + 1} u_1^{i_1+1} (1 - u_2 - u_1)^{i_0} \Big|_0^{1-u_2}$$

$$- \int_0^{1-u_2} \frac{1}{i_1 + 1} u_1^{i_1+1} d((1 - u_2 - u_1)^{i_0})$$

$$= - \int_0^{1-u_2} \frac{1}{i_1 + 1} u_1^{i_1+1} d((1 - u_2 - u_1)^{i_0})$$

$$= \int_0^{1-u_2} \frac{i_0}{i_1 + 1} u_1^{i_1+1} (1 - u_2 - u_1)^{i_0-1} du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0 - 1)}{(i_1 + 1)(i_1 + 2)} u_1^{i_1+2} (1 - u_2 - u_1)^{i_0-2} du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0 - 1)(i_0 - 2)}{(i_1 + 1)(i_1 + 2)(i_1 + 3)} u_1^{i_1+3} (1 - u_2 - u_1)^{i_0-3} du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0 - 1)(i_0 - 2)\ldots 2}{(i_1 + 1)(i_1 + 2)(i_1 + 3)\ldots(i_1 + i_0 - 1)} u_1^{i_1+i_0-1}$$
$$(1 - u_2 - u_1)^1 du_1$$

$$= \int_0^{1-u_2} \frac{i_0(i_0 - 1)(i_0 - 2)\ldots 1}{(i_1 + 1)(i_1 + 2)(i_1 + 3)\ldots(i_1 + i_0)} u_1^{i_1+i_0} du_1$$

$$= \int_0^{1-u_2} \frac{i_0! i_1!}{(i_1 + i_0)!} u_1^{i_1+i_0} du_1$$

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$$= \int_0^{1-u_2} \frac{i_0! i_1!}{(i_1 + i_0 + 1)!} d(u_1^{i_1+i_0+1})$$

$$= \frac{i_0! i_1!}{(i_0 + i_1 + 1)!} u_1^{i_0+i_1+1} \Big|_0^{1-u_2}$$

$$= \frac{i_0! i_1!}{(i_1 + i_0 + 1)!} (1 - u_2)^{i_1+i_0+1}$$

$$S_1 = \int_0^1 u_2^{i_2} du_2 \int_0^{1-u_2} u_1^{i_1} (1 - u_2 - u_1)^{i_0} du_1$$

$$= \int_0^1 u_2^{i_2} (1 - u_2)^{i_0+i_1+1} du_2 \frac{i_0! i_1!}{(i_0 + i_1 + 1)!}$$

$$= \frac{i_0! i_1! i_2!}{(i_0 + i_1 + i_2 + 2)!}$$

Based on the previous formula of S_1, the two dimensional integration of polynomials of degree 0, 1, 2 can be computed.

$$S_{P_0 P_1 P_2}(0,0) = sign(J) \int \int_{P_0 P_1 P_2} dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} du_1 du_2$$

$$S_{P_0 P_1 P_2}(0,0) = sign(J) \int \int_{P_0 P_1 P_2} dx_1 dx_2$$

$$= J \frac{0!}{(0+2)!} = \frac{1}{2} J$$

$$S_{P_0 P_1 P_2}(1,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1 dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} (x_{10} u_0 + x_{11} u_1 + x_{12} u_2) du_1 du_2$$

$$= J \frac{1!}{(1+2)!} (x_{10} + x_{11} + x_{12})$$

$$S_{P_0 P_1 P_2}(1,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1 dx_1 dx_2$$

$$= \frac{1}{6} J(x_{10} + x_{11} + x_{12})$$

$$S_{P_0 P_1 P_2}(0,1) = sign(J) \int \int_{P_0 P_1 P_2} x_2 dx_1 dx_2$$

$$= \frac{1}{6} J(x_{20} + x_{21} + x_{22})$$

$$S_{P_0 P_1 P_2}(2,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1^2 dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} (x_{10} u_0 + x_{11} u_1 + x_{12} u_2)^2 du_1 du_2$$

$$= J \frac{2!}{(2+2)!} (x_{10}^2 + x_{11}^2 + x_{12}^2)$$

$$+ J \frac{1!}{(2+2)!}$$

$$
\begin{array}{ccc}
(0 & +x_{10}x_{11} & +x_{10}x_{12} \\
+x_{11}x_{10} & +0 & +x_{11}x_{12} \\
+x_{12}x_{10} & +x_{12}x_{11} & +0)
\end{array}
$$

$$S_{P_0 P_1 P_2}(2,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1^2 dx_1 dx_2$$

$$= J \frac{1}{24}$$

$$
\begin{array}{ccc}
(2x_{10}x_{10} & +x_{10}x_{11} & +x_{10}x_{12} \\
+x_{11}x_{10} & +2x_{11}x_{11} & +x_{11}x_{12} \\
+x_{12}x_{10} & +x_{12}x_{11} & +2x_{12}x_{12})
\end{array}
$$

$$S_{P_0 P_1 P_2}(0,2) = sign(J) \int \int_{P_0 P_1 P_2} x_2^2 dx_1 dx_2$$

$$= J \frac{1}{24}$$

$$
\begin{array}{ccc}
(2x_{20}x_{20} & +x_{20}x_{21} & +x_{20}x_{22} \\
+x_{21}x_{20} & +2x_{21}x_{21} & +x_{21}x_{22} \\
+x_{22}x_{20} & +x_{22}x_{21} & +2x_{22}x_{22})
\end{array}
$$

$$S_{P_0 P_1 P_2}(1,1) = sign(J) \int \int_{P_0 P_1 P_2} x_1 x_2 dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} (x_{10} u_0 + x_{11} u_1 + x_{12} u_2)$$

$$(x_{20} u_0 + x_{21} u_1 + x_{22} u_2) du_1 du_2$$

$$= J \frac{2!}{(2+2)!} (x_{10}x_{20} + x_{11}x_{21} + x_{12}x_{22})$$

$$+ J \frac{1!}{(2+2)!}$$

$$
\begin{array}{ccc}
(0 & +x_{10}x_{21} & +x_{10}x_{22} \\
+x_{11}x_{20} & +0 & +x_{11}x_{22} \\
+x_{12}x_{20} & +x_{12}x_{21} & +0)
\end{array}
$$

$$S_{P_0 P_1 P_2}(1,1) = sign(J) \int \int_{P_0 P_1 P_2} x_1 x_2 dx_1 dx_2$$

$$= \frac{1}{24} J (2x_{10}x_{20} + 2x_{11}x_{21} + 2x_{12}x_{22}$$
$$+ \; x_{10}x_{21} + x_{10}x_{22} + x_{11}x_{20} + x_{11}x_{22} + x_{12}x_{20} + x_{12}x_{21})$$
$$= J \frac{1}{24}$$

$$
\begin{aligned}
(2x_{10}x_{20} &\quad + x_{10}x_{21} &\quad + x_{10}x_{22} \\
+ x_{11}x_{20} &\quad + 2x_{11}x_{21} &\quad + x_{11}x_{22} \\
+ x_{12}x_{20} &\quad + x_{12}x_{21} &\quad + 2x_{12}x_{22})
\end{aligned}
$$

Three Dimensional Simplex Integrations

The three dimensional integrations here in this section are used for three dimensional DDA algorithm. Since the displacement function is linear function of coordinates (x, y, z), the integrands are of degree 0, 1, 2.

A 3 dimensional simplex has 4 vertices

$$P_0, P_1, P_2, P_3.$$

$$
\begin{aligned}
P_0 &: \quad (x_{10}, \quad x_{20}, \quad x_{30}) \\
P_1 &: \quad (x_{11}, \quad x_{21}, \quad x_{31}) \\
P_2 &: \quad (x_{12}, \quad x_{22}, \quad x_{32}) \\
P_3 &: \quad (x_{13}, \quad x_{23}, \quad x_{33})
\end{aligned}
$$

The 3 dimensional coordinate simplex has 4 vertices

$$U_0, U_1, U_2, U_3.$$

$$
\begin{aligned}
U_0 &: \quad (\quad 0, \quad 0, \quad 0) \\
U_1 &: \quad (\quad 1, \quad 0, \quad 0) \\
U_2 &: \quad (\quad 0, \quad 1, \quad 0) \\
U_3 &: \quad (\quad 0, \quad 0, \quad 1)
\end{aligned}
$$

The following coordinate transformation

$$(u_1, u_2, u_3) \rightarrow (x_1, x_2, x_3)$$

s coordinate simplex

$$U_0 U_1 U_2 U_3$$

to normal simplex

$$P_0 P_1 P_2 P_3.$$

$$
\begin{array}{rclclclcl}
x_1 &=& x_{10} & (1 - \sum_1^3 u_i) + & x_{11} & u_1 & + & x_{12} & u_2 & + & x_{13} u_3 \\
x_2 &=& x_{20} & (1 - \sum_1^3 u_i) + & x_{21} & u_1 & + & x_{22} & u_2 & + & x_{23} u_3 \\
x_3 &=& x_{30} & (1 - \sum_1^3 u_i) + & x_{31} & u_1 & + & x_{32} & u_2 & + & x_{33} u_3
\end{array}
$$

The Jacobi determinant is

$$
J = \frac{D(x_1,\ x_2,\ x_3)}{D(u_1,\ u_2,\ u_3)}
$$

$$
= \begin{vmatrix}
\frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \frac{\partial x_1}{\partial u_3} \\
\frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_2}{\partial u_3} \\
\frac{\partial x_3}{\partial u_1} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_3}{\partial u_3}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
x_{11} - x_{10} & x_{12} - x_{10} & x_{13} - x_{10} \\
x_{21} - x_{20} & x_{22} - x_{20} & x_{23} - x_{20} \\
x_{31} - x_{30} & x_{32} - x_{30} & x_{33} - x_{30}
\end{vmatrix}
$$

$$
= \begin{vmatrix}
1 & x_{10} & x_{20} & x_{30} \\
1 & x_{11} & x_{21} & x_{31} \\
1 & x_{12} & x_{22} & x_{32} \\
1 & x_{13} & x_{23} & x_{33}
\end{vmatrix}
$$

Since $P_0 P_1 P_2 P_3$ is a 3-dimensional simplex with non-zero volume, J is non-zero. Translation can be rewritten as

$$
(u_0, u_1, u_2, u_3) \rightarrow (1, x_1, x_2, x_3)
$$

$$
\begin{array}{rclclclcl}
1 &=& & u_0 & + & & u_1 & + & & u_2 & + & & u_3 \\
x_1 &=& x_{10} & u_0 & + & x_{11} & u_1 & + & x_{12} & u_2 & + & x_{13} & u_3 \\
x_2 &=& x_{20} & u_0 & + & x_{21} & u_1 & + & x_{22} & u_2 & + & x_{23} & u_3 \\
x_3 &=& x_{30} & u_0 & + & x_{31} & u_1 & + & x_{32} & u_2 & + & x_{33} & u_3
\end{array}
$$

Three dimensional simplex integration

$$
S_{P_0 P_1 P_2 P_3}(m_1, m_2, m_3)
$$

on a n dimensional simplex

$$
P_0 P_1 P_2 P_3
$$

is defined as normal integration times the sign of determinant J.

$$
S_{P_0 P_1 P_2 P_3}(m_1, m_2, m_3)
$$

$$
= sign(J) \int \int \int_{P_0 P_1 P_2 P_3} x_1^{m_1} x_2^{m_2} x_3^{m_3}\ dx_1 dx_2 dx_3
$$

$$
= sign(J) \int \int \int_{U_0 U_1 U_2 U_3} x_1^{m_1} x_2^{m_2} x_3^{m_3} |J| du_1 du_2 du_3
$$

$$
= J \int \int \int_{U_0 U_1 U_2 U_3} x_1^{m_1} x_2^{m_2} x_3^{m_3}\ du_1 du_2 du_3
$$

Here

$$x_l^{m_l} = \left(\sum_{k=0}^{3} x_{lk} u_k\right)^{m_l}$$

then the three dimensional simplex integration can be represented by the following basic forms of simplex integration:

$$
\begin{aligned}
S_1 &= \int \int \int_{U_0 U_1 U_2 U_3} u_0^{i_0} u_1^{i_1} u_2^{i_2} u_3^{i_3} du_1 du_2 du_3 \\
&= \int \int \int_{u_0,u_1,u_2,u_3 \geq 0}^{u_0+u_1+u_2+u_3=1} u_0^{i_0} u_1^{i_1} u_2^{i_2} u_3^{i_3} du_1 du_2 du_3 \\
&= \int_0^1 \int_0^{1-u_3} \int_0^{1-u_3-u_2} u_3^{i_3} u_2^{i_2} u_1^{i_1} u_0^{i_0} du_1 du_2 du_3 \\
&= \int_0^1 \int_0^{1-u_3} u_3^{i_3} u_2^{i_2} \left(\int_0^{1-u_3-u_2} u_1^{i_1} (1 - u_3 - u_2 - u_1)^{i_0} du_1\right) du_2 du_3
\end{aligned}
$$

After i_1 times of integration by parts, the inner integration can be computed.

$$
\begin{aligned}
&\int_0^{1-u_3-u_2} u_1^{i_1} (1 - u_3 - u_2 - u_1)^{i_0} du_1 \\
&= \int_0^{1-u_3-u_2} d(\frac{1}{i_1+1} u_1^{i_1+1})(1 - u_3 - u_2 - u_1)^{i_0} \\
&= \frac{1}{i_1+1} u_1^{i_1+1}(1 - u_3 - u_2 - u_1)^{i_0} \Big|_0^{1-u_3-u_2} \\
&\quad - \int_0^{1-u_3-u_2} \frac{1}{i_1+1} u_1^{i_1+1} d((1 - u_3 - u_2 - u_1)^{i_0}) \\
&= - \int_0^{1-u_3-u_2} \frac{1}{i_1+1} u_1^{i_1+1} d((1 - u_3 - u_2 - u_1)^{i_0}) \\
&= \int_0^{1-u_3-u_2} \frac{i_0}{i_1+1} u_1^{i_1+1}(1 - u_3 - u_2 - u_1)^{i_0-1} du_1 \\
&= \int_0^{1-u_3-u_2} \frac{i_0(i_0-1)}{(i_1+1)(i_1+2)} u_1^{i_1+2}(1 - u_3 - u_2 - u_1)^{i_0-2} du_1 \\
&= \int_0^{1-u_3-u_2} \frac{i_0(i_0-1)(i_0-2)}{(i_1+1)(i_1+2)(i_1+3)} u_1^{i_1+3}(1 - u_3 - u_2 - u_1)^{i_0-3} du_1 \\
&= \int_0^{1-u_3-u_2} \frac{i_0(i_0-1)(i_0-2)\ldots 2}{(i_1+1)(i_1+2)(i_1+3)\ldots(i_1+i_0-1)} u_1^{i_1+i_0-1} \\
&\quad (1 - u_3 - u_2 - u_1)^1 du_1 \\
&= \int_0^{1-u_3-u_2} \frac{i_0(i_0-1)(i_0-2)\ldots 1}{(i_1+1)(i_1+2)(i_1+3)\ldots(i_1+i_0)} u_1^{i_1+i_0} du_1
\end{aligned}
$$

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$$= \int_0^{1-u_3-u_2} \frac{i_0! i_1!}{(i_1+i_0)!} u_1^{i_1+i_0} du_1$$

$$= \int_0^{1-u_3-u_2} \frac{i_0! i_1!}{(i_1+i_0+1)!} d(u_1^{i_1+i_0+1})$$

$$= \frac{i_0! i_1!}{(i_0+i_1+1)!} u_1^{i_0+i_1+1} \Big|_0^{1-u_3-u_2}$$

$$= \frac{i_0! i_1!}{(i_1+i_0+1)!} (1-u_3-u_2)^{i_1+i_0+1}$$

$$S_1 = \int_0^1 \int_0^{1-u_3} u_3^{i_3} u_2^{i_2} du_2 du_3 \int_0^{1-u_3-u_2} u_1^{i_1} (1-u_3-u_2-u_1)^{i_0} du_1$$

$$= \int_0^1 u_3^{i_3} \int_0^{1-u_3} u_2^{i_2} (1-u_3-u_2)^{i_0+i_1+1} du_2 du_3 \frac{i_0! i_1!}{(i_0+i_1+1)!}$$

$$= \int_0^1 u_3^{i_3} (1-u_3)^{i_0+i_1+i_2+2} du_3 \frac{i_0! i_1! i_2!}{(i_0+i_1+i_2+2)!}$$

$$= \frac{i_0! i_1! i_2! i_3!}{(i_0+i_1+i_2+i_3+3)!}$$

Based on the previous formula of S_1, the three dimensional integration of polynomials of degree 0, 1, 2 can be computed.

$$S_{P_0 P_1 P_2 P_3}(0,0,0) = sign(J) \int \int \int_{P_0 P_1 P_2 P_3} dx_1 dx_2 dx_3$$

$$= J \frac{0!}{(0+3)!} = \frac{1}{6} J$$

$$S_{P_0 P_1 P_2 P_3}(1,0,0) = sign(J) \int \int \int_{P_0 P_1 P_2 P_3} x_1 dx_1 dx_2 dx_3$$

$$= J \int \int \int_{U_0 U_1 U_2 U_3} (x_{10} u_0 + x_{11} u_1 + x_{12} u_2 + x_{13} u_3)$$

$$du_1 du_2 du_3$$

$$= J \frac{1!}{(1+3)!} (x_{10} + x_{11} + x_{12} + x_{13})$$

$$S_{P_0 P_1 P_2 P_3}(1,0,0) = sign(J) \int \int \int_{P_0 P_1 P_2 P_3} x_1 dx_1 dx_2 dx_3$$

$$= J \frac{1}{24} (x_{10} + x_{11} + x_{12} + x_{13})$$

$$S_{P_0 P_1 P_2 P_3}(0,1,0) = sign(J) \int \int \int_{P_0 P_1 P_2 P_3} x_2 dx_1 dx_2 dx_3$$

$$= J\frac{1}{24}(x_{20} + x_{21} + x_{22} + x_{23})$$

$$S_{P_0P_1P_2P_3}(0,0,1) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_3\,dx_1dx_2dx_3$$

$$= J\frac{1}{24}(x_{30} + x_{31} + x_{32} + x_{33})$$

$$S_{P_0P_1P_2P_3}(2,0,0) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_1^2\,dx_1dx_2dx_3$$

$$= J\int\int\int_{U_0U_1U_2U_3} (x_{10}u_0 + x_{11}u_1 + x_{12}u_2 + x_{13}u_3)^2$$
$$du_1\,du_2\,du_3$$

$$= J\frac{2!}{(2+3)!}(x_{10}^2 + x_{11}^2 + x_{12}^2 + x_{13}^2)$$

$$+ J\frac{1!}{(2+3)!}$$

$$\begin{array}{llll}
(0 & +x_{10}x_{11} & +x_{10}x_{12} & +x_{10}x_{13} \\
+x_{11}x_{10} & +0 & +x_{11}x_{12} & +x_{11}x_{13} \\
+x_{12}x_{10} & +x_{12}x_{11} & +0 & +x_{12}x_{13} \\
+x_{13}x_{10} & +x_{13}x_{11} & +x_{13}x_{12} & +0)
\end{array}$$

$$S_{P_0P_1P_2P_3}(2,0,0) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_1^2\,dx_1dx_2dx_3$$

$$= J\frac{1}{120}$$

$$\begin{array}{llll}
(2x_{10}x_{10} & +x_{10}x_{11} & +x_{10}x_{12} & +x_{10}x_{13} \\
+x_{11}x_{10} & +2x_{11}x_{11} & +x_{11}x_{12} & +x_{11}x_{13} \\
+x_{12}x_{10} & +x_{12}x_{11} & +2x_{12}x_{12} & +x_{12}x_{13} \\
+x_{13}x_{10} & +x_{13}x_{11} & +x_{13}x_{12} & +2x_{13}x_{13})
\end{array}$$

$$S_{P_0P_1P_2P_3}(0,2,0) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_2^2\,dx_1dx_2dx_3$$

$$= J\frac{1}{120}$$

$$\begin{array}{llll}
(2x_{20}x_{20} & +x_{20}x_{21} & +x_{20}x_{22} & +x_{20}x_{23} \\
+x_{21}x_{20} & +2x_{21}x_{21} & +x_{21}x_{22} & +x_{21}x_{23} \\
+x_{22}x_{20} & +x_{22}x_{21} & +2x_{22}x_{22} & +x_{22}x_{23} \\
+x_{23}x_{20} & +x_{23}x_{21} & +x_{23}x_{22} & +2x_{23}x_{23})
\end{array}$$

$$S_{P_0P_1P_2P_3}(0,0,2) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_3^2\,dx_1dx_2dx_3$$

$$= J\frac{1}{120}$$

$$(2x_{30}x_{30} \quad +x_{30}x_{31} \quad +x_{30}x_{32} \quad +x_{30}x_{33}$$
$$+x_{31}x_{30} \quad +2x_{31}x_{31} \quad +x_{31}x_{32} \quad +x_{31}x_{33}$$
$$+x_{32}x_{30} \quad +x_{32}x_{31} \quad +2x_{32}x_{32} \quad +x_{32}x_{33}$$
$$+x_{33}x_{30} \quad +x_{33}x_{31} \quad +x_{33}x_{32} \quad +2x_{33}x_{33})$$

$$S_{P_0P_1P_2P_3}(1,1,0) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_1 x_2 \, dx_1 dx_2 dx_3$$

$$= J \int\int\int_{U_0U_1U_2U_3} (x_{10}u_0 + x_{11}u_1 + x_{12}u_2 + x_{13}u_3)$$

$$(x_{20}u_0 + x_{21}u_1 + x_{22}u_2 + x_{23}u_3) du_1 du_2 du_3$$

$$= J\frac{2!}{(2+3)!}(x_{10}x_{20} + x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23})$$

$$+ J\frac{1!}{(2+3)!}$$

$$(0 \quad +x_{10}x_{21} \quad +x_{10}x_{22} \quad +x_{10}x_{23}$$
$$+x_{11}x_{20} \quad +0 \quad +x_{11}x_{22} \quad +x_{11}x_{23}$$
$$+x_{12}x_{20} \quad +x_{12}x_{21} \quad +0 \quad +x_{12}x_{23}$$
$$+x_{13}x_{20} \quad +x_{13}x_{21} \quad +x_{13}x_{22} \quad +0)$$

$$S_{P_0P_1P_2P_3}(1,1,0) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_1 x_2 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$(2x_{10}x_{20} \quad +x_{10}x_{21} \quad +x_{10}x_{22} \quad +x_{10}x_{23}$$
$$+x_{11}x_{20} \quad +2x_{11}x_{21} \quad +x_{11}x_{22} \quad +x_{11}x_{23}$$
$$+x_{12}x_{20} \quad +x_{12}x_{21} \quad +2x_{12}x_{22} \quad +x_{12}x_{23}$$
$$+x_{13}x_{20} \quad +x_{13}x_{21} \quad +x_{13}x_{22} \quad +2x_{13}x_{23})$$

$$S_{P_0P_1P_2P_3}(1,0,1) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_1 x_3 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$(2x_{10}x_{30} \quad +x_{10}x_{31} \quad +x_{10}x_{32} \quad +x_{10}x_{33}$$
$$+x_{11}x_{30} \quad +2x_{11}x_{31} \quad +x_{11}x_{32} \quad +x_{11}x_{33}$$
$$+x_{12}x_{30} \quad +x_{12}x_{31} \quad +2x_{12}x_{32} \quad +x_{12}x_{33}$$
$$+x_{13}x_{30} \quad +x_{13}x_{31} \quad +x_{13}x_{32} \quad +2x_{13}x_{33})$$

$$S_{P_0P_1P_2P_3}(0,1,1) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_2 x_3 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$(2x_{20}x_{30} \quad +x_{20}x_{31} \quad +x_{20}x_{32} \quad +x_{20}x_{33}$$
$$+x_{21}x_{30} \quad +2x_{21}x_{31} \quad +x_{21}x_{32} \quad +x_{21}x_{33}$$
$$+x_{22}x_{30} \quad +x_{22}x_{31} \quad +2x_{22}x_{32} \quad +x_{22}x_{33}$$
$$+x_{23}x_{30} \quad +x_{23}x_{31} \quad +x_{23}x_{32} \quad +2x_{23}x_{33})$$

Three Dimensional Simplex Integrations to the Coordinate Simplex

A n dimensional simplex has n+1 vertices

$$P_0, P_1, P_2, \ldots, P_n.$$

$$
\begin{array}{llllll}
P_0 : & (x_{10}, & x_{20}, & x_{30}, & \ldots, & x_{n0}) \\
P_1 : & (x_{11}, & x_{21}, & x_{31}, & \ldots, & x_{n1}) \\
P_2 : & (x_{12}, & x_{22}, & x_{32}, & \ldots, & x_{n2}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
P_n : & (x_{1n}, & x_{2n}, & x_{3n}, & \ldots, & x_{nn})
\end{array}
$$

The n dimensional coordinate simplex has n+1 vertices

$$U_0, U_1, U_2, \ldots, U_n.$$

$$
\begin{array}{llllll}
U_0 : & (\ 0, & 0, & 0, & \ldots, & 0 \ ) \\
U_1 : & (\ 1, & 0, & 0, & \ldots, & 0 \ ) \\
U_2 : & (\ 0, & 1, & 0, & \ldots, & 0 \ ) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
U_n : & (\ 0, & 0, & 0, & \ldots, & 1 \ )
\end{array}
\tag{1}
$$

The following coordinate transformation

$$(u_1, u_2, u_3, \ldots, u_n) \to (x_1, x_2, x_3, \ldots, x_n)$$

transfers coordinate simplex

$$U_0 U_1 U_2 \ldots U_n$$

to normal simplex

$$P_0 P_1 P_2 \ldots P_n.$$

$$
\begin{array}{llllllll}
x_1 & = & x_{10} & (1 - \sum_1^n u_i) + & x_{11} \ u_1 & + & x_{12} \ u_2 + \ldots & + & x_{1n} u_n \\
x_2 & = & x_{20} & (1 - \sum_1^n u_i) + & x_{21} \ u_1 & + & x_{22} \ u_2 + \ldots & + & x_{2n} u_n \\
x_3 & = & x_{30} & (1 - \sum_1^n u_i) + & x_{31} \ u_1 & + & x_{32} \ u_2 + \ldots & + & x_{3n} u_n \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
x_n & = & x_{n0} & (1 - \sum_1^n u_i) + & x_{n1} \ u_1 & + & x_{n2} \ u_2 + \ldots & + & x_{nn} u_n
\end{array}
\tag{2}
$$

The Jacobi determinant is

$$J = \frac{D(x_1, \ x_2, \ x_3, \ \ldots, \ x_n)}{D(u_1, \ u_2, \ u_3, \ \ldots, \ u_n)}$$

$$= \begin{vmatrix} \dfrac{\partial x_1}{\partial u_1} & \dfrac{\partial x_1}{\partial u_2} & \dfrac{\partial x_1}{\partial u_3} & \cdots & \dfrac{\partial x_1}{\partial u_n} \\[2mm] \dfrac{\partial x_2}{\partial u_1} & \dfrac{\partial x_2}{\partial u_2} & \dfrac{\partial x_2}{\partial u_3} & \cdots & \dfrac{\partial x_2}{\partial u_n} \\[2mm] \dfrac{\partial x_3}{\partial u_1} & \dfrac{\partial x_3}{\partial u_2} & \dfrac{\partial x_3}{\partial u_3} & \cdots & \dfrac{\partial x_3}{\partial u_n} \\[2mm] \vdots & \vdots & \vdots & \vdots & \vdots \\[1mm] \dfrac{\partial x_n}{\partial u_1} & \dfrac{\partial x_n}{\partial u_2} & \dfrac{\partial x_n}{\partial u_3} & \cdots & \dfrac{\partial x_n}{\partial u_n} \end{vmatrix}$$

$$= \begin{vmatrix} x_{11} - x_{10} & x_{12} - x_{10} & x_{13} - x_{10} & \cdots & x_{1n} - x_{10} \\ x_{21} - x_{20} & x_{22} - x_{20} & x_{23} - x_{20} & \cdots & x_{2n} - x_{20} \\ x_{31} - x_{30} & x_{32} - x_{30} & x_{33} - x_{30} & \cdots & x_{3n} - x_{30} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n1} - x_{n0} & x_{n2} - x_{n0} & x_{n3} - x_{n0} & \cdots & x_{nn} - x_{n0} \end{vmatrix}$$

$$= \begin{vmatrix} x_{11} - x_{10} & x_{21} - x_{20} & x_{31} - x_{30} & \cdots & x_{n1} - x_{n0} \\ x_{12} - x_{10} & x_{22} - x_{20} & x_{32} - x_{30} & \cdots & x_{n2} - x_{n0} \\ x_{13} - x_{10} & x_{23} - x_{20} & x_{33} - x_{30} & \cdots & x_{n3} - x_{n0} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{1n} - x_{10} & x_{2n} - x_{20} & x_{3n} - x_{30} & \cdots & x_{nn} - x_{n0} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x_{10} & x_{20} & x_{30} & \cdots & x_{n0} \\ 1 & x_{11} & x_{21} & x_{31} & \cdots & x_{n1} \\ 1 & x_{12} & x_{22} & x_{32} & \cdots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} & x_{3n} & \cdots & x_{nn} \end{vmatrix} \tag{3}$$

Since $P_0 P_1 P_2 \ldots P_n$ is a n-dimensional simplex with non-zero volume, J is non-zero. Translation (2) can be rewritten as

$$(u_0, u_1, u_2, u_3, \ldots, u_n) \to (1, x_1, x_2, x_3, \ldots, x_n)$$

$$\begin{aligned}
1 &= & u_0 & + & & u_1 & + & & u_2 & + \ldots + & & u_n \\
x_1 &= & x_{10}\, u_0 & + & x_{11}\, & u_1 & + & x_{12}\, & u_2 & + \ldots + & x_{1n}\, & u_n \\
x_2 &= & x_{20}\, u_0 & + & x_{21}\, & u_1 & + & x_{22}\, & u_2 & + \ldots + & x_{2n}\, & u_n \\
x_3 &= & x_{30}\, u_0 & + & x_{31}\, & u_1 & + & x_{32}\, & u_2 & + \ldots + & x_{3n}\, & u_n \\
\vdots & & \vdots & & & \vdots & & & \vdots & \vdots & & \vdots \\
x_n &= & x_{n0}\, u_0 & + & x_{n1}\, & u_1 & + & x_{n2}\, & u_2 & + \ldots + & x_{nn}\, & u_n
\end{aligned}$$

$$\begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_{10} & x_{11} & x_{12} & \cdots & x_{1n} \\ x_{20} & x_{21} & x_{22} & \cdots & x_{2n} \\ x_{30} & x_{31} & x_{32} & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{n0} & x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{pmatrix} \tag{4}$$

The determinant is

$$
\begin{vmatrix}
1 & 1 & 1 & \cdots & 1 \\
x_{10} & x_{11} & x_{12} & \cdots & x_{1n} \\
x_{20} & x_{21} & x_{22} & \cdots & x_{2n} \\
x_{30} & x_{31} & x_{32} & \cdots & x_{3n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n0} & x_{n1} & x_{n2} & \cdots & x_{nn}
\end{vmatrix}
=
\begin{vmatrix}
1 & x_{10} & x_{20} & x_{30} & \cdots & x_{n0} \\
1 & x_{11} & x_{21} & x_{31} & \cdots & x_{n1} \\
1 & x_{12} & x_{22} & x_{32} & \cdots & x_{n2} \\
1 & x_{13} & x_{23} & x_{33} & \cdots & x_{n3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & x_{1n} & x_{2n} & x_{3n} & \cdots & x_{nn}
\end{vmatrix}
= J \qquad (5)
$$

From formula (4) $(u_0, u_1, u_2, u_3, \ldots, u_n)$ can be computed

$$
\begin{pmatrix}
u_0 \\ u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n
\end{pmatrix}
=
\begin{pmatrix}
1 & 1 & 1 & \cdots & 1 \\
x_{10} & x_{11} & x_{12} & \cdots & x_{1n} \\
x_{20} & x_{21} & x_{22} & \cdots & x_{2n} \\
x_{30} & x_{31} & x_{32} & \cdots & x_{3n} \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
x_{n0} & x_{n1} & x_{n2} & \cdots & x_{nn}
\end{pmatrix}^{-1}
\begin{pmatrix}
1 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n
\end{pmatrix}
$$

Definition of N Dimensional Simplex Integration

Simplex integration

$$
S_{P_0 P_1 P_2 \ldots P_n}(m_1, m_2, m_3, \ldots, m_n)
$$

on a n dimensional simplex

$$
P_0 P_1 P_2 \ldots P_n
$$

is defined as normal integration times the sign of determinant J.

$$
\begin{aligned}
S_{P_0 P_1 P_2 \ldots P_n}&(m_1, m_2, m_3, \ldots, m_n) \\
&= sign(J) \int \int \int \cdots \int_{P_0 P_1 P_2 \ldots P_n} x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n} \; dx_1 dx_2 dx_3 \ldots dx_n \\
&= sign(J) \int \int \int \cdots \int_{U_0 U_1 U_2 \ldots U_n} x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n} |J| du_1 du_2 du_3 \ldots du_n \\
&= \quad J \int \int \int \cdots \int_{U_0 U_1 U_2 \ldots U_n} x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_n^{m_n} \; du_1 du_2 du_3 \ldots du_n \qquad (6)
\end{aligned}
$$

In order to compute the integration, coefficients of invariants u_k^i in

$$
x_l^{m_l} = \Big(\sum_{k=0}^{n} x_{lk} u_k \Big)^{m_l}
$$

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of formula (6), the following formula (7) is useful.

The formula

$$\left(\sum_{k=0}^{n} a_k\right)^m = \sum_{\substack{j_0+j_1+j_2+\ldots+j_n=m \\ j_0,j_1,j_2,\ldots,j_n \geq 0}} \frac{(j_0 + j_1 + j_2 + \ldots + j_n)!}{j_0!j_1!j_2!\ldots!j_n!} a_0^{j_0} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} \qquad (7)$$

can be found in algebra books, however a brief derivation are given here for convenience.
In case of n=0, formula (7) is correct:

$$\left(\sum_{k=0}^{0} a_k\right)^m = \sum_{\substack{j_0=m \\ j_0 \geq 0}} \frac{j_0!}{j_0!} a_0^{j_0} = a_0^m$$

In case of n=1, formula (7) is correct:

$$\left(\sum_{k=0}^{1} a_k\right)^m = \sum_{\substack{j_0+j_1=m \\ j_0,j_1 \geq 0}} \frac{(j_0 + j_1)!}{j_0!j_1!} a_0^{j_0} a_1^{j_1}$$

Assuming formula (7) is correct for n:

$$\left(\sum_{k=0}^{n} a_k\right)^m = \sum_{\substack{j_0+j_1+j_2+\ldots+j_n=m \\ j_0,j_1,j_2,\ldots,j_n \geq 0}} \frac{(j_0 + j_1 + j_2 + \ldots + j_n)!}{j_0!j_1!j_2!\ldots!j_n!} a_0^{j_0} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n}$$

then the following computation shows formula (7) is correct for n+1

$$\left(\sum_{k=0}^{n+1} a_k\right)^m$$

$$=\left(\left(\sum_{k=0}^{n} a_k\right) + a_{n+1}\right)^m$$

$$= \sum_{\substack{i_n+j_{n+1}=m \\ i_n,j_{n+1} \geq 0}} \frac{(i_n + j_{n+1})!}{i_n!j_{n+1}!} \left(\sum_{k=0}^{n} a_k\right)^{i_n} a_{n+1}^{j_{n+1}}$$

$$= \sum_{\substack{i_n+j_{n+1}=m \\ i_n,j_{n+1} \geq 0}} \frac{(i_n + j_{n+1})!}{i_n!j_{n+1}!}$$

$$\sum_{\substack{j_0+j_1+j_2+\ldots+j_n=i_n \\ j_0,j_1,j_2,\ldots,j_n \geq 0}} \frac{(j_0 + j_1 + j_2 + \ldots + j_n)!}{j_0!j_1!j_2!\ldots!j_n!} a_0^{j_0} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} a_{n+1}^{j_{n+1}}$$

$$= \sum_{\substack{i_n+j_{n+1}=m \\ i_n,j_{n+1} \geq 0}} \sum_{\substack{j_0+j_1+j_2+\ldots+j_n=i_n \\ j_0,j_1,j_2,\ldots,j_n \geq 0}}$$

$$\frac{(i_n + j_{n+1})!}{i_n!j_{n+1}!} \frac{(j_0 + j_1 + j_2 + \ldots + j_n)!}{j_0!j_1!j_2!\ldots!j_n!} a_0^{j_0} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} a_{n+1}^{j_{n+1}}$$

$$= \sum_{\substack{i_n,j_{n+1}\geq 0}}^{i_n+j_{n+1}=m} \sum_{\substack{j_0,j_1,j_2,\ldots,j_n\geq 0}}^{j_0+j_1+j_2+\ldots+j_n=i_n}$$

$$\frac{(i_n+j_{n+1})!}{j_0!j_1!j_2!\ldots!j_n!j_{n+1}!} a_0^{j_0} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} a_{n+1}^{j_{n+1}}$$

$$= \sum_{\substack{j_0,j_1,j_2,\ldots,j_n,j_{n+1}\geq 0}}^{j_0+j_1+j_2+\ldots+j_n+j_{n+1}=m}$$

$$\frac{(j_0+j_1+j_2+\ldots+j_n+j_{n+1})!}{j_0!j_1!j_2!\ldots!j_n!j_{n+1}!} a_0^{j_0} a_1^{j_1} a_2^{j_2} \cdots a_n^{j_n} a_{n+1}^{j_{n+1}}$$

Denote

$$u_0 = 1 - \sum_1^n u_i,$$

from the previous algebraic formula,

$$x_l^{m_l} = \left(\sum_{k=0}^n u_k x_{lk}\right)^{m_l}$$

$$= \sum_{\substack{i_{l0},i_{l1},i_{l2},\ldots,i_{ln}\geq 0}}^{i_{l0}+i_{l1}+i_{l2}+\ldots+i_{ln}=m_l} \frac{(i_{l0}+i_{l1}+i_{l2}+\ldots+i_{ln})!}{i_{l0}!i_{l1}!i_{l2}!\ldots!i_{ln}!}$$

$$u_0^{i_{l0}} u_1^{i_{l1}} u_2^{i_{l2}} \cdots u_n^{i_{ln}} x_{l0}^{i_{l0}} x_{l1}^{i_{l1}} x_{l2}^{i_{l2}} \cdots x_{ln}^{i_{ln}} \qquad (8)$$

Substituting (8) into (6), integral (9) is obtained.

$$S_{P_0P_1P_2\ldots P_n}(m_1,m_2,m_3,\ldots,m_n) = J \int \int \int \cdots \int_{U_0U_1U_2\ldots U_n}$$

$$\left(\sum_{\substack{i_{10},i_{11},i_{12},\ldots,i_{1n}\geq 0}}^{i_{10}+i_{11}+i_{12}+\ldots+i_{1n}=m_1} \frac{(i_{10}+i_{11}+i_{12}+\ldots+i_{1n})!}{i_{10}!i_{11}!i_{12}!\ldots!i_{1n}!}\right.$$

$$\left. u_0^{i_{10}} u_1^{i_{11}} u_2^{i_{12}} \cdots u_n^{i_{1n}} \quad x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} \cdots x_{1n}^{i_{1n}}\right)$$

$$\left(\sum_{\substack{i_{20},i_{21},i_{22},\ldots,i_{2n}\geq 0}}^{i_{20}+i_{21}+i_{22}+\ldots+i_{2n}=m_2} \frac{(i_{20}+i_{21}+i_{22}+\ldots+i_{2n})!}{i_{20}!i_{21}!i_{22}!\ldots!i_{2n}!}\right.$$

$$\left. u_0^{i_{20}} u_1^{i_{21}} u_2^{i_{22}} \cdots u_n^{i_{2n}} \quad x_{20}^{i_{20}} x_{21}^{i_{21}} x_{22}^{i_{22}} \cdots x_{2n}^{i_{2n}}\right)$$

$$\left(\sum_{\substack{i_{30},i_{31},i_{32},\ldots,i_{3n}\geq 0}}^{i_{30}+i_{31}+i_{32}+\ldots+i_{3n}=m_3} \frac{(i_{30}+i_{31}+i_{32}+\ldots+i_{3n})!}{i_{30}!i_{31}!i_{32}!\ldots!i_{3n}!}\right.$$

$$\left. u_0^{i_{30}} u_1^{i_{31}} u_2^{i_{32}} \cdots u_n^{i_{3n}} \quad x_{30}^{i_{30}} x_{31}^{i_{31}} x_{32}^{i_{32}} \cdots x_{3n}^{i_{3n}}\right)$$

$$\ldots \ldots \ldots \ldots \ldots \ldots \ldots$$

$$\left(\sum_{\substack{i_{n0},i_{n1},i_{n2},\ldots,i_{nn}\geq 0}}^{i_{n0}+i_{n1}+i_{n2}+\ldots+i_{nn}=m_n} \frac{(i_{n0}+i_{n1}+i_{n2}+\ldots+i_{nn})!}{i_{n0}!i_{n1}!i_{n2}!\ldots!i_{nn}!}\right.$$

$$\left. u_0^{i_{n0}} u_1^{i_{n1}} u_2^{i_{n2}} \cdots u_n^{i_{nn}} \quad x_{n0}^{i_{n0}} x_{n1}^{i_{n1}} x_{n2}^{i_{n2}} \cdots x_{nn}^{i_{nn}}\right)$$

$$du_1 du_2 du_3 \ldots du_n \qquad (9)$$

$$S_{P_0 P_1 P_2 \ldots P_n}(m_1, m_2, m_3, \ldots, m_n)$$

$$= J \sum_{\substack{i_{10}, i_{11}, i_{12}, \ldots, i_{1n} \geq 0}}^{i_{10}+i_{11}+i_{12}+\ldots+i_{1n}=m_1} \sum_{\substack{i_{20}, i_{21}, i_{22}, \ldots, i_{2n} \geq 0}}^{i_{20}+i_{21}+i_{22}+\ldots+i_{2n}=m_2}$$

$$\sum_{\substack{i_{30}, i_{31}, i_{32}, \ldots, i_{3n} \geq 0}}^{i_{30}+i_{31}+i_{32}+\ldots+i_{3n}=m_3} \cdots \sum_{\substack{i_{n0}, i_{n1}, i_{n2}, \ldots, i_{nn} \geq 0}}^{i_{n0}+i_{n1}+i_{n2}+\ldots+i_{nn}=m_n}$$

$$\frac{m_1!}{i_{10}! i_{11}! i_{12}! \ldots ! i_{1n}!} \qquad \frac{m_2!}{i_{20}! i_{21}! i_{22}! \ldots ! i_{2n}!}$$

$$\frac{m_3!}{i_{30}! i_{31}! i_{32}! \ldots ! i_{3n}!} \quad \cdots \quad \frac{m_n!}{i_{n0}! i_{n1}! i_{n2}! \ldots ! i_{nn}!}$$

$$x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} \cdots x_{1n}^{i_{1n}} \qquad x_{20}^{i_{20}} x_{21}^{i_{21}} x_{22}^{i_{22}} \cdots x_{2n}^{i_{2n}}$$

$$x_{30}^{i_{30}} x_{31}^{i_{31}} x_{32}^{i_{32}} \cdots x_{3n}^{i_{3n}} \quad \cdots \quad x_{n0}^{i_{n0}} x_{n1}^{i_{n1}} x_{n2}^{i_{n2}} \cdots x_{nn}^{i_{nn}}$$

$$\int \int \int \cdots \int_{U_0 U_1 U_2 \ldots U_n} u_0^{i_0} u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n} \, du_1 du_2 du_3 \ldots du_n \qquad (10)$$

where

$$i_l = i_{1l} + i_{2l} + i_{3l} + \cdots + i_{nl}, \quad l = 0, 1, 2, \ldots, n.$$

Formula of N Dimensional Simplex Integration

The coordinate simplex integration in (10) is another form of well known Dirichlet integration in classical analysis. The direct computation of the integration is given below.

$$S_1 = \int \int \int \cdots \int_{U_0 U_1 U_2 \ldots U_n} u_0^{i_0} u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n} \, du_1 du_2 du_3 \ldots du_n$$

$$= \int \int \int \cdots \int_{\substack{u_0, u_1, u_2, \ldots, u_n \geq 0}}^{u_0+u_1+u_2+\ldots+u_n=1} u_0^{i_0} u_1^{i_1} u_2^{i_2} \cdots u_n^{i_n} \, du_1 du_2 du_3 \ldots du_n$$

$$= \int_0^1 \int_0^{1-u_n} \int_0^{1-u_n-u_{n-1}} \cdots \int_0^{1-u_n-u_{n-1}-\cdots-u_2}$$

$$u_n^{i_n} u_{n-1}^{i_{n-1}} u_{n-2}^{i_{n-2}} \cdots u_1^{i_1} u_0^{i_0} \, du_1 du_2 du_3 \ldots du_n$$

$$= \int_0^1 \int_0^{1-u_n} \int_0^{1-u_n-u_{n-1}} \cdots \int_0^{1-u_n-u_{n-1}-\cdots-u_3}$$

$$u_n^{i_n} u_{n-1}^{i_{n-1}} u_{n-2}^{i_{n-2}} \cdots u_2^{i_2}$$

$$\left(\int_0^{1-u_n-u_{n-1}-\cdots-u_2} u_1^{i_1} \left(1 - \sum_{k=1}^n u_k \right)^{i_0} du_1 \right) du_2 du_3 \ldots du_n \qquad (11)$$

After i_1 times of integration by parts, the inner integration of (11) can be computed.

$$\int_0^{1-u_n-u_{n-1}-\cdots-u_2} u_1^{i_1} \left(1 - \sum_{k=1}^n u_k \right)^{i_0} du_1$$

$$= \int_0^{1-u_n-u_{n-1}-\cdots-u_2} d(\frac{1}{i_1+1}u_1^{i_1+1})\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0}$$

$$=\frac{1}{i_1+1}u_1^{i_1+1}\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0}\Big|_0^{1-u_n-u_{n-1}-\cdots-u_2}$$

$$-\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{1}{i_1+1}u_1^{i_1+1}d(\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0})$$

$$=-\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{1}{i_1+1}u_1^{i_1+1}d(\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0})$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0}{i_1+1}u_1^{i_1+1}\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0-1}du_1$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0(i_0-1)}{(i_1+1)(i_1+2)}u_1^{i_1+2}\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0-2}du_1$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0(i_0-1)(i_0-2)}{(i_1+1)(i_1+2)(i_1+3)}u_1^{i_1+3}\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0-3}du_1$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0(i_0-1)(i_0-2)\ldots 2}{(i_1+1)(i_1+2)(i_1+3)\ldots(i_1+i_0-1)}u_1^{i_1+i_0-1}$$

$$\left(1-\sum\nolimits_{k=1}^n u_k\right)^1 du_1$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0(i_0-1)(i_0-2)\ldots 1}{(i_1+1)(i_1+2)(i_1+3)\ldots(i_1+i_0)}u_1^{i_1+i_0}du_1$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0!i_1!}{(i_1+i_0)!}u_1^{i_1+i_0}du_1$$

$$=\int_0^{1-u_n-u_{n-1}-\cdots-u_2}\frac{i_0!i_1!}{(i_1+i_0+1)!}d(u_1^{i_1+i_0+1})$$

$$=\frac{i_0!i_1!}{(i_0+i_1+1)!}u_1^{i_0+i_1+1}\Big|_0^{1-u_n-u_{n-1}-\cdots-u_2}$$

$$=\frac{i_0!i_1!}{(i_1+i_0+1)!}\left(1-\sum\nolimits_{k=2}^n u_k\right)^{i_1+i_0+1} \tag{12}$$

$$S_1=\int_0^1\int_0^{1-u_n}\int_0^{1-u_n-u_{n-1}}\cdots\int_0^{1-u_n-u_{n-1}-\cdots-u_3}u_n^{i_n}u_{n-1}^{i_{n-1}}u_{n-2}^{i_{n-2}}\cdots u_2^{i_2}$$

$$\int_0^{1-u_n-u_{n-1}-\cdots-u_2}u_1^{i_1}\left(1-\sum\nolimits_{k=1}^n u_k\right)^{i_0}du_1 du_2 du_3\ldots du_n$$

$$=\int_0^1\int_0^{1-u_n}\int_0^{1-u_n-u_{n-1}}\cdots\int_0^{1-u_n-u_{n-1}-\cdots-u_4}u_n^{i_n}u_{n-1}^{i_{n-1}}u_{n-2}^{i_{n-2}}\cdots u_3^{i_3}$$

$$\int_0^{1-u_n-u_{n-1}-\cdots-u_3}u_2^{i_2}(1-\sum\nolimits_{k=2}^n u_k)^{i_0+i_1+1}$$

$$du_2 du_3 du_4\ldots du_n\frac{i_0!i_1!}{(i_0+i_1+1)!}$$

$$= \int_0^1 \int_0^{1-u_n} \int_0^{1-u_n-u_{n-1}} \cdots \int_0^{1-u_n-u_{n-1}-\cdots-u_4} u_n^{i_n} u_{n-1}^{i_{n-1}} u_{n-2}^{i_{n-2}} \cdots u_4^{i_4} u_3^{i_3}$$

$$\left(1 - \sum_{k=3}^n u_k\right)^{i_0+i_1+i_2+2} du_3 du_4 du_5 \ldots du_n \frac{i_0! i_1! i_2!}{(i_0 + i_1 + i_2 + 2)!}$$

$$= \int_0^1 \int_0^{1-u_n} \int_0^{1-u_n-u_{n-1}} \cdots \int_0^{1-u_n-u_{n-1}-\cdots-u_5} u_n^{i_n} u_{n-1}^{i_{n-1}} u_{n-2}^{i_{n-2}} \cdots u_4^{i_4}$$

$$\int_0^{1-u_n-u_{n-1}-\cdots-u_4} u_3^{i_3} \left(1 - \sum_{k=3}^n u_k\right)^{i_0+i_1+i_2+2}$$

$$du_3 du_4 du_5 \ldots du_n \frac{i_0! i_1! i_2!}{(i_0 + i_1 + i_2 + 2)!}$$

$$= \int_0^1 \int_0^{1-u_n} u_n^{i_n} u_{n-1}^{i_{n-1}} \left(1 - u_{n-1} - u_n\right)^{i_0+i_1+i_2+\cdots+i_{n-2}+n-2} du_{n-1} du_n$$

$$\frac{i_0! i_1! i_2! \ldots . ! i_{n-2}!}{(i_0 + i_1 + i_2 + \ldots + i_{n-2} + n - 2)!}$$

$$= \int_0^1 u_n^{i_n} \left(1 - u_n\right)^{i_0+i_1+i_2+\cdots+i_{n-1}+n-1} du_n$$

$$\frac{i_0! i_1! i_2! \ldots . ! i_{n-1}!}{(i_0 + i_1 + i_2 + \ldots + i_{n-1} + n - 1)!}$$

$$= \frac{i_0! i_1! i_2! \ldots . ! i_n!}{(i_0 + i_1 + i_2 + \ldots + i_n + n)!} \tag{13}$$

Substituting equation (13) into (11), the integrals of a n dimensional simplex

$$S_{P_0 P_1 P_2 \ldots P_n}(m_1, m_2, m_3, \ldots, m_n)$$

is obtained.

$$S_{P_0 P_1 P_2 \ldots P_n}(m_1, m_2, m_3, \ldots, m_n)$$

$$= J \sum_{i_{10},i_{11},i_{12},\ldots,i_{1n} \geq 0}^{i_{10}+i_{11}+i_{12}+\ldots+i_{1n}=m_1} \quad \sum_{i_{20},i_{21},i_{22},\ldots,i_{2n} \geq 0}^{i_{20}+i_{21}+i_{22}+\ldots+i_{2n}=m_2}$$

$$\sum_{i_{30},i_{31},i_{32},\ldots,i_{3n} \geq 0}^{i_{30}+i_{31}+i_{32}+\ldots+i_{3n}=m_3} \quad \cdots \quad \sum_{i_{n0},i_{n1},i_{n2},\ldots,i_{nn} \geq 0}^{i_{n0}+i_{n1}+i_{n2}+\ldots+i_{nn}=m_n}$$

$$\frac{m_1!}{i_{10}! i_{11}! i_{12}! \ldots ! i_{1n}!} \quad \frac{m_2!}{i_{20}! i_{21}! i_{22}! \ldots ! i_{2n}!}$$

$$\frac{m_3!}{i_{30}! i_{31}! i_{32}! \ldots ! i_{3n}!} \quad \cdots \quad \frac{m_n!}{i_{n0}! i_{n1}! i_{n2}! \ldots ! i_{nn}!}$$

$$x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} \cdots x_{1n}^{i_{1n}} \quad x_{20}^{i_{20}} x_{21}^{i_{21}} x_{22}^{i_{22}} \cdots x_{2n}^{i_{2n}}$$

$$x_{30}^{i_{30}} x_{31}^{i_{31}} x_{32}^{i_{32}} \cdots x_{3n}^{i_{3n}} \quad \cdots \quad x_{n0}^{i_{n0}} x_{n1}^{i_{n1}} x_{n2}^{i_{n2}} \cdots x_{nn}^{i_{nn}}$$

$$\frac{i_0! i_1! i_2! \ldots ! i_n!}{(i_0 + i_1 + i_2 + \ldots + i_n + n)!} \tag{14}$$

where

$$i_l = i_{1l} + i_{2l} + i_{3l} + \cdots + i_{nl}, \quad l = 0, 1, 2, \ldots, n.$$

Examples of Simplex Integrations on Simplex

A general n dimensional simplex has n+1 ordered vertices $P_0 P_1 P_2 \ldots P_n$, its Jacobian is

$$J = \begin{vmatrix} 1 & x_{10} & x_{20} & x_{30} & \cdots & x_{n0} \\ 1 & x_{11} & x_{21} & x_{31} & \cdots & x_{n1} \\ 1 & x_{12} & x_{22} & x_{32} & \cdots & x_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{1n} & x_{2n} & x_{3n} & \cdots & x_{nn} \end{vmatrix} \tag{15}$$

Formula (14) directly gives many integrations on a simplex.

$$S_{P_0 P_1 P_2 \ldots P_n}(0, \ldots, 0, m_l, 0, \ldots, 0)$$

$$= \int \int \int \cdots \int_{P_0 P_1 P_2 \ldots P_n} x_l^{m_l} \; du_1 du_2 du_3 \ldots du_n$$

$$= J \sum_{i_{l0}, i_{l1}, i_{l2}, \ldots, i_{ln} \geq 0}^{i_{l0} + i_{l1} + i_{l2} + \ldots + i_{ln} = m_l} \frac{m_l!}{i_{l0}! i_{l1}! i_{l2}! \ldots! i_{ln}!}$$

$$x_{l0}^{i_{l0}} x_{l1}^{i_{l1}} x_{l2}^{i_{l2}} \cdots x_{ln}^{i_{ln}} \frac{i_0! i_1! i_2! \ldots! i_n!}{(i_0 + i_1 + i_2 + \ldots + i_n + n)!}$$

$$= J \sum_{i_{l0}, i_{l1}, i_{l2}, \ldots, i_{ln} \geq 0}^{i_{l0} + i_{l1} + i_{l2} + \ldots + i_{ln} = m_l} x_{l0}^{i_{l0}} x_{l1}^{i_{l1}} x_{l2}^{i_{l2}} \cdots x_{ln}^{i_{ln}} \frac{m_l!}{(m_l + n)!}$$

As special case, the normal one dimensional integral formula (16) can be derived.

$$S_{P_0 P_1}(m) = sign(J) \int_{P_0 P_1} x_1^m dx_1 = J \sum_{i_{10}, i_{11} \geq 0}^{i_{10} + i_{11} = m} x_{10}^{i_{10}} x_{11}^{i_{11}} \frac{m!}{(m+1)!}$$

$$= \frac{1}{(m+1)}(x_{11} - x_{10})$$

$$(x_{11}^m x_{10}^0 + x_{11}^{m-1} x_{10}^1 + x_{11}^{m-2} x_{10}^2 + \ldots + x_{11}^1 x_{10}^{m-1} + x_{11}^0 x_{10}^m)$$

$$= \frac{1}{m+1}(x_{11}^{m+1} - x_{10}^{m+1}) \tag{16}$$

$$S_{P_0 P_1 P_2}(0, 0) = sign(J) \int \int_{P_0 P_1 P_2} dx_1 dx_2$$

$$= J \int \int_{U_0 U_1 U_2} du_1 du_2$$

$$= J \sum_{i_{10}, i_{11}, i_{12} \geq 0}^{i_{10}+i_{11}+i_{12}=0} \frac{0!}{(0+2)!}$$

$$S_{P_0 P_1 P_2}(0,0) = sign(J) \int \int_{P_0 P_1 P_2} dx_1 dx_2$$

$$= \frac{1}{2} J \tag{17}$$

$$S_{P_0 P_1 P_2}(1,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1 dx_1 dx_2$$

$$= J \sum_{i_{10}, i_{11}, i_{12} \geq 0}^{i_{10}+i_{11}+i_{12}=1} \frac{1!}{(1+2)!} x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}}$$

$$S_{P_0 P_1 P_2}(1,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1 dx_1 dx_2$$

$$= \frac{1}{6} J(x_{10} + x_{11} + x_{12})$$

$$S_{P_0 P_1 P_2}(0,1) = sign(J) \int \int_{P_0 P_1 P_2} x_2 dx_1 dx_2$$

$$= \frac{1}{6} J(x_{20} + x_{21} + x_{22}) \tag{18}$$

$$S_{P_0 P_1 P_2}(2,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1^2 dx_1 dx_2$$

$$= J \sum_{i_{10}, i_{11}, i_{12} \geq 0}^{i_{10}+i_{11}+i_{12}=2} \frac{2!}{(2+2)!} x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}}$$

$$S_{P_0 P_1 P_2}(2,0) = sign(J) \int \int_{P_0 P_1 P_2} x_1^2 dx_1 dx_2$$

$$= \frac{1}{12} J(x_{10}^2 + x_{11}^2 + x_{12}^2 + x_{10}x_{11} + x_{10}x_{12} + x_{11}x_{12})$$

$$= J\frac{1}{24}$$

$$
\begin{aligned}
(2x_{10}x_{10} &\quad +x_{10}x_{11} &\quad +x_{10}x_{12} \\
+x_{11}x_{10} &\quad +2x_{11}x_{11} &\quad +x_{11}x_{12} \\
+x_{12}x_{10} &\quad +x_{12}x_{11} &\quad +2x_{12}x_{12})
\end{aligned}
$$

$$S_{P_0 P_1 P_2}(0,2) = sign(J) \int \int_{P_0 P_1 P_2} x_2^2 dx_1 dx_2$$

$$= \frac{1}{12} J(x_{20}^2 + x_{21}^2 + x_{22}^2 + x_{20}x_{21} + x_{20}x_{22} + x_{21}x_{22})$$

$$= J\frac{1}{24}$$

$$\begin{aligned}
(2x_{20}x_{20} &\quad +x_{20}x_{21} &\quad +x_{20}x_{22} \\
+x_{21}x_{20} &\quad +2x_{21}x_{21} &\quad +x_{21}x_{22} \\
+x_{22}x_{20} &\quad +x_{22}x_{21} &\quad +2x_{22}x_{22})
\end{aligned} \tag{19}$$

$$S_{P_0P_1P_2}(1,1) = sign(J) \int\int_{P_0P_1P_2} x_1 x_2 \, dx_1 dx_2$$

$$= J \sum_{i_{10},i_{11},i_{12}\geq 0}^{i_{10}+i_{11}+i_{12}=1} \sum_{i_{20},i_{21},i_{22}\geq 0}^{i_{20}+i_{21}+i_{22}=1}$$

$$\frac{1}{1} x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} \frac{1}{1} x_{20}^{i_{20}} x_{21}^{i_{21}} x_{22}^{i_{22}} \frac{(i_{10}+i_{20})!(i_{11}+i_{21})!(i_{12}+i_{22})!}{(2+2)!}$$

$$S_{P_0P_1P_2}(1,1) = sign(J) \int\int_{P_0P_1P_2} x_1 x_2 \, dx_1 dx_2$$

$$= \frac{1}{24} J(2x_{10}x_{20} + 2x_{11}x_{21} + 2x_{12}x_{22}$$

$$+ x_{10}x_{21} + x_{10}x_{22} + x_{11}x_{20} + x_{11}x_{22} + x_{12}x_{20} + x_{12}x_{21})$$

$$= J \frac{1}{24}$$

$$\begin{aligned}
(2x_{10}x_{20} &\quad +x_{10}x_{21} &\quad +x_{10}x_{22} \\
+x_{11}x_{20} &\quad +2x_{11}x_{21} &\quad +x_{11}x_{22} \\
+x_{12}x_{20} &\quad +x_{12}x_{21} &\quad +2x_{12}x_{22})
\end{aligned} \tag{20}$$

$$S_{P_0P_1P_2P_3}(0,0,0) = sign(J) \int\int\int_{P_0P_1P_2P_3} dx_1 dx_2 dx_3$$

$$= J \frac{0!}{(0+3)!} = \frac{1}{6} J \tag{21}$$

$$S_{P_0P_1P_2P_3}(1,0,0) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_1 \, dx_1 dx_2 dx_3$$

$$= J \sum_{i_{10},i_{11},i_{12},i_{13}\geq 0}^{i_{10}+i_{11}+i_{12}+i_{13}=1} x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} x_{13}^{i_{13}} \frac{1!}{(1+3)!}$$

$$S_{P_0P_1P_2P_3}(1,0,0) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_1 \, dx_1 dx_2 dx_3$$

$$= J \frac{1}{24}(x_{10} + x_{11} + x_{12} + x_{13})$$

$$S_{P_0P_1P_2P_3}(0,1,0) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_2 \, dx_1 dx_2 dx_3$$

$$= J \frac{1}{24}(x_{20} + x_{21} + x_{22} + x_{23})$$

$$S_{P_0P_1P_2P_3}(0,0,1) = sign(J) \int\int\int_{P_0P_1P_2P_3} x_3 \, dx_1 dx_2 dx_3$$

$$= J \frac{1}{24}(x_{30} + x_{31} + x_{32} + x_{33}) \tag{22}$$

$$S_{P_0P_1P_2P_3}(2,0,0) = sign(J) \int \int \int_{P_0P_1P_2P_3} x_1^2 \, dx_1 dx_2 dx_3$$

$$= J \sum_{i_{10},i_{11},i_{12},i_{13}\geq 0}^{i_{10}+i_{11}+i_{12}+i_{13}=2} x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} x_{13}^{i_{13}} \frac{2!}{(2+3)!}$$

$$= J\frac{1}{60}(x_{10}^2 + x_{11}^2 + x_{12}^2 + x_{13}^2 + x_{10}x_{11} + x_{10}x_{12} + x_{10}x_{13}$$

$$+ x_{11}x_{12} + x_{11}x_{13} + x_{12}x_{13})$$

$$S_{P_0P_1P_2P_3}(2,0,0) = sign(J) \int \int \int_{P_0P_1P_2P_3} x_1^2 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$
\begin{array}{llll}
(2x_{10}x_{10} & +x_{10}x_{11} & +x_{10}x_{12} & +x_{10}x_{13} \\
+x_{11}x_{10} & +2x_{11}x_{11} & +x_{11}x_{12} & +x_{11}x_{13} \\
+x_{12}x_{10} & +x_{12}x_{11} & +2x_{12}x_{12} & +x_{12}x_{13} \\
+x_{13}x_{10} & +x_{13}x_{11} & +x_{13}x_{12} & +2x_{13}x_{13})
\end{array}
$$

$$S_{P_0P_1P_2P_3}(0,2,0) = sign(J) \int \int \int_{P_0P_1P_2P_3} x_2^2 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$
\begin{array}{llll}
(2x_{20}x_{20} & +x_{20}x_{21} & +x_{20}x_{22} & +x_{20}x_{23} \\
+x_{21}x_{20} & +2x_{21}x_{21} & +x_{21}x_{22} & +x_{21}x_{23} \\
+x_{22}x_{20} & +x_{22}x_{21} & +2x_{22}x_{22} & +x_{22}x_{23} \\
+x_{23}x_{20} & +x_{23}x_{21} & +x_{23}x_{22} & +2x_{23}x_{23})
\end{array}
$$

$$S_{P_0P_1P_2P_3}(0,0,2) = sign(J) \int \int \int_{P_0P_1P_2P_3} x_3^2 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$
\begin{array}{llll}
(2x_{30}x_{30} & +x_{30}x_{31} & +x_{30}x_{32} & +x_{30}x_{33} \\
+x_{31}x_{30} & +2x_{31}x_{31} & +x_{31}x_{32} & +x_{31}x_{33} \\
+x_{32}x_{30} & +x_{32}x_{31} & +2x_{32}x_{32} & +x_{32}x_{33} \\
+x_{33}x_{30} & +x_{33}x_{31} & +x_{33}x_{32} & +2x_{33}x_{33})
\end{array}
$$

(23)

$$S_{P_0P_1P_2P_3}(1,1,0) = sign(J) \int \int \int_{P_0P_1P_2P_3} x_1 x_2 \, dx_1 dx_2 dx_3$$

$$= J \sum_{i_{10},i_{11},i_{12},i_{13}\geq 0}^{i_{10}+i_{11}+i_{12}+i_{13}=1} \sum_{i_{20},i_{21},i_{22},i_{23}\geq 0}^{i_{20}+i_{21}+i_{22}+i_{23}=1}$$

$$x_{10}^{i_{10}} x_{11}^{i_{11}} x_{12}^{i_{12}} x_{13}^{i_{13}} x_{20}^{i_{20}} x_{21}^{i_{21}} x_{22}^{i_{22}} x_{23}^{i_{23}}$$

$$\frac{(i_{10}+i_{20})!(i_{11}+i_{21})!(i_{12}+i_{22})!(i_{13}+i_{23})!}{(2+3)!}$$

$$S_{P_0P_1P_2P_3}(1,1,0) = sign(J) \int \int \int_{P_0P_1P_2P_3} x_1 x_2 \, dx_1 dx_2 dx_3$$

$$= J\frac{1}{120}$$

$$
\begin{array}{llll}
(2x_{10}x_{20} & +x_{10}x_{21} & +x_{10}x_{22} & +x_{10}x_{23} \\
+x_{11}x_{20} & +2x_{11}x_{21} & +x_{11}x_{22} & +x_{11}x_{23} \\
+x_{12}x_{20} & +x_{12}x_{21} & +2x_{12}x_{22} & +x_{12}x_{23} \\
+x_{13}x_{20} & +x_{13}x_{21} & +x_{13}x_{22} & +2x_{13}x_{23})
\end{array}
$$

$$S_{P_0P_1P_2P_3}(1,0,1) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_1 x_3\, dx_1\, dx_2\, dx_3$$

$$= J\frac{1}{120}$$

$$
\begin{array}{llll}
(2x_{10}x_{30} & +x_{10}x_{31} & +x_{10}x_{32} & +x_{10}x_{33} \\
+x_{11}x_{30} & +2x_{11}x_{31} & +x_{11}x_{32} & +x_{11}x_{33} \\
+x_{12}x_{30} & +x_{12}x_{31} & +2x_{12}x_{32} & +x_{12}x_{33} \\
+x_{13}x_{30} & +x_{13}x_{31} & +x_{13}x_{32} & +2x_{13}x_{33})
\end{array}
$$

$$S_{P_0P_1P_2P_3}(0,1,1) = sign(J)\int\int\int_{P_0P_1P_2P_3} x_2 x_3\, dx_1\, dx_2\, dx_3$$

$$= J\frac{1}{120}$$

$$
\begin{array}{llll}
(2x_{20}x_{30} & +x_{20}x_{31} & +x_{20}x_{32} & +x_{20}x_{33} \\
+x_{21}x_{30} & +2x_{21}x_{31} & +x_{21}x_{32} & +x_{21}x_{33} \\
+x_{22}x_{30} & +x_{22}x_{31} & +2x_{22}x_{32} & +x_{22}x_{33} \\
+x_{23}x_{30} & +x_{23}x_{31} & +x_{23}x_{32} & +2x_{23}x_{33})
\end{array}
\tag{24}
$$

One dimensional simplex integration (16) is same as conventional integration. Formulae (17)-(20) are 2-d simplex integrations over triangles. Formulae (21)-(24) are 3-d simplex integrations over tetrahedrons.

Simplex Integration on General Two-dimensional Blocks

Since simplex integrations always have the Jacobian J as factor and J is an oriented area, the integrations on positive area and negative area can be neutralized. Denote

$$P_1 P_2 P_3 \cdots P_n \qquad P_{n+1} = P_1$$

with

$$P_i = (x_{1i}, x_{2i})$$

as a polygon, the integrations (17)-(20) on

$$P_1 P_2 P_3 \cdots P_n$$

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are computed by summations. (25)-(28) are the integrations of DDA method. Here P_0 can be any point, $P_0 = (0,0)$ are chosen in order to have simpler formulae.

$$\iint_{(A)} dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(0,0)$$

$$= \frac{1}{2} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix} \tag{25}$$

$$\iint_{(A)} x_1 dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(1,0)$$

$$\iint_{(A)} x_1 dx_1 dx_2 = \frac{1}{6} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix} (x_{1\,k} + x_{1\,k+1})$$

$$\iint_{(A)} x_2 dx_1 dx_2 = \frac{1}{6} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix} (x_{2\,k} + x_{2\,k+1}) \tag{26}$$

$$\iint_{(A)} x_1^2 dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(2,0)$$

$$= \frac{1}{12} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(x_{1\,k}^2 + x_{1\,k+1}^2 + x_{1\,k} x_{1\,k+1})$$

$$\iint_{(A)} x_1^2 dx_1 dx_2 = \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(2x_{1\,k} x_{1\,k} \quad + x_{1\,k} x_{1\,k+1}$$
$$+ x_{1\,k+1} x_{1\,k} \quad + 2x_{1\,k+1} x_{1\,k+1})$$

$$\iint_{(A)} x_2^2 dx_1 dx_2 = \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(2x_{2\,k} x_{2\,k} \quad + x_{2\,k} x_{2\,k+1}$$
$$+ x_{2\,k+1} x_{2\,k} \quad + 2x_{2\,k+1} x_{2\,k+1}) \tag{27}$$

$$\iint_{(A)} x_1 x_2 dx_1 dx_2 = \sum_{k=1}^{n} S_{P_0 P_k P_{k+1}}(1,1)$$

$$= \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(2x_{1\,k} x_{2\,k} + 2x_{1\,k+1} x_{2\,k+1} + x_{1\,k} x_{2\,k+1} + x_{1\,k+1} x_{2\,k})$$

$$\iint_{(A)} x_1 x_2 dx_1 dx_2 = \frac{1}{24} \sum_{k=1}^{n} \begin{vmatrix} x_{1\,k} & x_{2\,k} \\ x_{1\,k+1} & x_{2\,k+1} \end{vmatrix}$$
$$(2x_{1\,k} x_{2\,k} \quad + x_{1\,k} x_{2\,k+1}$$
$$+ x_{1\,k+1} x_{2\,k} \quad + 2x_{1\,k+1} x_{2\,k+1}) \tag{28}$$

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Figure 3. shows equal-lateral 2-dimensional polygons. The equal-lateral triangle, square, pentagon and hexagon are in the unit circle. The distance from each node to the circle center is 1. The edge length and area of each equal-lateral polygon are listed in the following table. The centers of gravity of the equal-lateral triangle, square, pentagon and hexagon are the center of the unit circle.
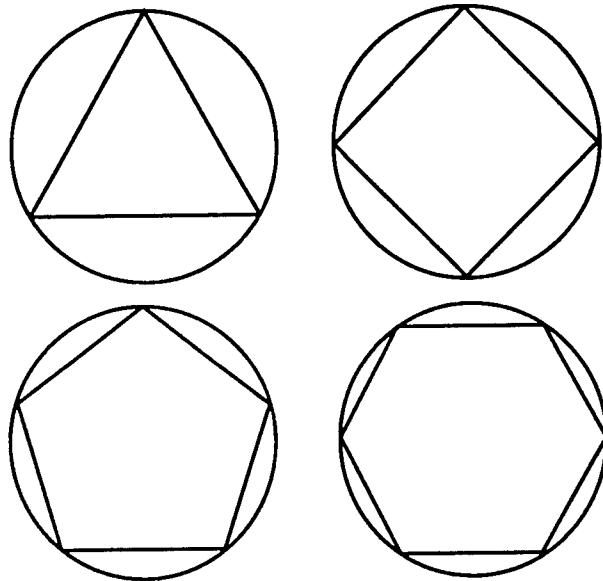
Figure 3. Equal-lateral of 2-dimensional polygons

The coordinates (x_i, y_i) of vertices of equal-lateral polygon can be computed by the following formulae:

$$x_i = sin(360/n * i)$$
$$y_i = cos(360/n * i)$$
$$i = 1, 2, \cdots, n$$

here n is the edge number of the polygons.

Using the simplex integrations, the same area and center of gravity are obtained for each equal-lateral polygon.

polygon	edge length	area
equal-lateral triangle	$\sqrt{3}$	$\frac{1}{4}3\sqrt{3} = 1.299038$
equal-lateral square	$\sqrt{2}$	2.000000
equal-lateral pentagon	$\sqrt{\frac{1}{2}(5-\sqrt{5})}$	$\frac{5}{8}\sqrt{10+2\sqrt{5}} = 2.377641$
equal-lateral hexagon	1	$\frac{1}{2}3\sqrt{3} = 2.598076$

Simplex Integrations on General Three-dimensional Blocks

Integrations (21)-(24) are the integrations of tetrahedrons. By summations of simplex integrations (21)-(24), the volume or integrations of any 3-d block can be computed. Assume

$$P_1^{[i]} P_2^{[i]} P_3^{[i]} \cdots P_{n(i)}^{[i]}$$

$$P_{n(i)+1}^{[i]} = P_1^{[i]} \qquad i = 1,2,3,\ldots,s$$

are all outward rotated polygons of a block,

$$P_j^{[i]} = (x_{1j}^{[i]}, x_{2j}^{[i]}, x_{2j}^{[i]})$$

and $P_0 = (0,0,0)$. The volume of this block is given by (29). Computed by simplex integrations, integrals (29)-(32) are represented by the coordinates of the boundary vertices only.

$$\iiint_{(V)} dx_1\,dx_2\,dx_3 = \sum_{i=1}^{s} \sum_{k=1}^{n(i)} S_{P_0 P_1^{[i]} P_k^{[i]} P_{k+1}^{[i]}}(0,0,0)$$

$$= \frac{1}{6} \sum_{i=1}^{s} \sum_{k=1}^{n(i)} \begin{vmatrix} x_{1\,1}^{[i]} & x_{2\,1}^{[i]} & x_{3\,1}^{[i]} \\ x_{1\,k}^{[i]} & x_{2\,k}^{[i]} & x_{3\,k}^{[i]} \\ x_{1\,k+1}^{[i]} & x_{2\,k+1}^{[i]} & x_{3\,k+1}^{[i]} \end{vmatrix} \qquad (29)$$

$$\iiint_{(V)} x_1\,dx_1\,dx_2\,dx_3 = \sum_{i=1}^{s} \sum_{k=1}^{n(i)} S_{P_0 P_1^{[i]} P_k^{[i]} P_{k+1}^{[i]}}(1,0,0)$$

$$\iiint_{(V)} x_1\,dx_1\,dx_2\,dx_3 = \frac{1}{24} \sum_{i=1}^{s} \sum_{k=1}^{n(i)}$$

$$\begin{vmatrix} x_{1\,1}^{[i]} & x_{2\,1}^{[i]} & x_{3\,1}^{[i]} \\ x_{1\,k}^{[i]} & x_{2\,k}^{[i]} & x_{3\,k}^{[i]} \\ x_{1\,k+1}^{[i]} & x_{2\,k+1}^{[i]} & x_{3\,k+1}^{[i]} \end{vmatrix}$$

$$\iiint_{(V)} x_2 dx_1 dx_2 dx_3 = \frac{1}{24} \sum_{i=1}^{s} \sum_{k=1}^{n(i)} \begin{vmatrix} x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\ x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\ x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]} \end{vmatrix}$$

$$(x_{1\ 1} + x_{1\ k} + x_{1\ k+1})$$

$$\iiint_{(V)} x_3 dx_1 dx_2 dx_3 = \frac{1}{24} \sum_{i=1}^{s} \sum_{k=1}^{n(i)} \begin{vmatrix} x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\ x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\ x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]} \end{vmatrix}$$

$$(x_{2\ 1} + x_{2\ k} + x_{2\ k+1})$$

$$(x_{3\ 1} + x_{3\ k} + x_{3\ k+1}) \tag{30}$$

$$\iiint_{(V)} x_1^2 dx_1 dx_2 dx_3 = \sum_{i=1}^{s} \sum_{k=1}^{n(i)} S_{P_0 P_1^{[i]} P_k^{[i]} P_{k+1}^{[i]}}(2,0,0)$$

$$\iiint_{(V)} x_1^2 dx_1 dx_2 dx_3 = \frac{1}{120} \sum_{i=1}^{s} \sum_{k=1}^{n(i)} \begin{vmatrix} x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\ x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\ x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]} \end{vmatrix}$$

$$\begin{pmatrix} 2x_{1\ 1}x_{1\ 1} & +x_{1\ 1}x_{1\ k} & +x_{1\ 1}x_{1\ k+1} \\ +x_{1\ k}x_{1\ 1} & +2x_{1\ k}x_{1\ k} & +x_{1\ k}x_{1\ k+1} \\ +x_{1\ k+1}x_{1\ 1} & +x_{1\ k+1}x_{1\ k} & +2x_{1\ k+1}x_{1\ k+1} \end{pmatrix}$$

$$\iiint_{(V)} x_2^2 dx_1 dx_2 dx_3 = \frac{1}{120} \sum_{i=1}^{s} \sum_{k=1}^{n(i)} \begin{vmatrix} x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\ x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\ x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]} \end{vmatrix}$$

$$\begin{pmatrix} 2x_{2\ 1}x_{2\ 1} & +x_{2\ 1}x_{2\ k} & +x_{2\ 1}x_{2\ k+1} \\ +x_{2\ k}x_{2\ 1} & +2x_{2\ k}x_{2\ k} & +x_{2\ k}x_{2\ k+1} \\ +x_{2\ k+1}x_{2\ 1} & +x_{2\ k+1}x_{2\ k} & +2x_{2\ k+1}x_{2\ k+1} \end{pmatrix}$$

$$\iiint_{(V)} x_3^2 dx_1 dx_2 dx_3 = \frac{1}{120} \sum_{i=1}^{s} \sum_{k=1}^{n(i)} \begin{vmatrix} x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\ x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\ x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]} \end{vmatrix}$$

$$
\begin{pmatrix}
2x_{3\ 1}x_{3\ 1} & +x_{3\ 1}x_{3\ k} & +x_{3\ 1}x_{3\ k+1} \\
+x_{3\ k}x_{3\ 1} & +2x_{3\ k}x_{3\ k} & +x_{3\ k}x_{3\ k+1} \\
+x_{3\ k+1}x_{3\ 1} & +x_{3\ k+1}x_{3\ k} & +2x_{3\ k+1}x_{3\ k+1}
\end{pmatrix}
\tag{31}
$$

$$
\int\int\int_{(V)} x_1 x_2\, dx_1 dx_2 dx_3 = \sum_{i=1}^{s}\sum_{k=1}^{n(i)} S_{P_0 P_1^{[i]} P_k^{[i]} P_{k+1}^{[i]}}(1,1,0)
$$

$$
\int\int\int_{(V)} x_1 x_2\, dx_1 dx_2 dx_3 = \frac{1}{120}\sum_{i=1}^{s}\sum_{k=1}^{n(i)}
$$

$$
\begin{vmatrix}
x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\
x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\
x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]}
\end{vmatrix}
$$

$$
\begin{pmatrix}
2x_{1\ 1}x_{2\ 1} & +x_{1\ 1}x_{2\ k} & +x_{1\ 1}x_{2\ k+1} \\
+x_{1\ k}x_{2\ 1} & +2x_{1\ k}x_{2\ k} & +x_{1\ k}x_{2\ k+1} \\
+x_{1\ k+1}x_{2\ 1} & +x_{1\ k+1}x_{2\ k} & +2x_{1\ k+1}x_{2\ k+1}
\end{pmatrix}
$$

$$
\int\int\int_{(V)} x_1 x_3\, dx_1 dx_2 dx_3 = \frac{1}{120}\sum_{i=1}^{s}\sum_{k=1}^{n(i)}
$$

$$
\begin{vmatrix}
x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\
x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\
x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]}
\end{vmatrix}
$$

$$
\begin{pmatrix}
2x_{1\ 1}x_{3\ 1} & +x_{1\ 1}x_{3\ k} & +x_{1\ 1}x_{3\ k+1} \\
+x_{1\ k}x_{3\ 1} & +2x_{1\ k}x_{3\ k} & +x_{1\ k}x_{3\ k+1} \\
+x_{1\ k+1}x_{3\ 1} & +x_{1\ k+1}x_{3\ k} & +2x_{1\ k+1}x_{3\ k+1}
\end{pmatrix}
$$

$$
\int\int\int_{(V)} x_2 x_3\, dx_1 dx_2 dx_3 = \frac{1}{120}\sum_{i=1}^{s}\sum_{k=1}^{n(i)}
$$

$$
\begin{vmatrix}
x_{1\ 1}^{[i]} & x_{2\ 1}^{[i]} & x_{3\ 1}^{[i]} \\
x_{1\ k}^{[i]} & x_{2\ k}^{[i]} & x_{3\ k}^{[i]} \\
x_{1\ k+1}^{[i]} & x_{2\ k+1}^{[i]} & x_{3\ k+1}^{[i]}
\end{vmatrix}
$$

$$
\begin{pmatrix}
2x_{2\ 1}x_{3\ 1} & +x_{2\ 1}x_{3\ k} & +x_{2\ 1}x_{3\ k+1} \\
+x_{2\ k}x_{3\ 1} & +2x_{2\ k}x_{3\ k} & +x_{2\ k}x_{3\ k+1} \\
+x_{2\ k+1}x_{3\ 1} & +x_{2\ k+1}x_{3\ k} & +2x_{2\ k+1}x_{3\ k+1}
\end{pmatrix}
\tag{32}
$$

Figure 4. shows 3 different blocks. Each block is shown from two different view angles. For each block, the geometric formulae of boundary plan angle, edge length, distance of boundary polygon node to the polygon center, distance of polygon plane to the unit sphere center, and the theoretical volume of the block are listed in the following tables.

The first block is a tetrahedron with four equal-lateral triangles as its boundary faces. The block is in the unit sphere. The distance from each vertex to the sphere center is 1.
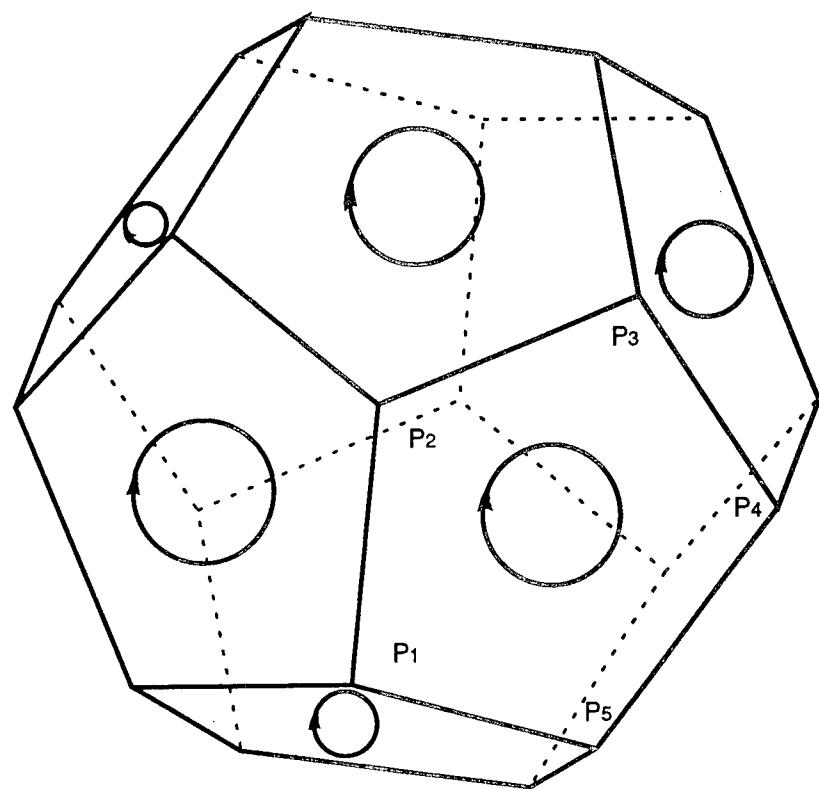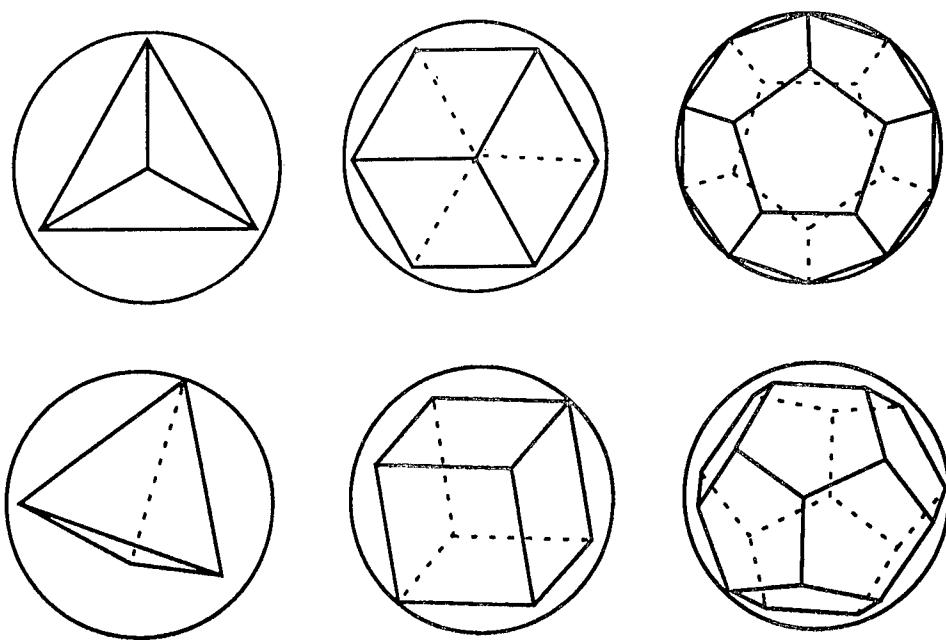
Figure 4. 3-dimensional blocks with equal-lateral boundary polygon

First block of 4 face polygons

boundary polygon number	4
edge number of polygons	3
angle of two adjacent polygon planes	$arctan(2\sqrt{2})$
edge length of polygons	$2\sqrt{\frac{2}{3}}$
distance from polygon to sphere center	$\frac{1}{3}$
distance of node to its polygon center	$\frac{2}{3}\sqrt{2}$
distance from vertex to sphere center	1
volume of block	$\frac{8}{27}\sqrt{3} = 0.51320$

The second block is a cube with six equal-lateral squares as its boundary faces. The block is in the unit sphere. The distance from each vertex to the sphere center is 1.

Second block of 6 face polygons

boundary polygon number	6
edge number of polygons	4
angle of two adjacent polygon planes	90
edge length of polygons	$\frac{2}{\sqrt{3}}$
distance from polygon to sphere center	$\frac{1}{\sqrt{3}}$
distance of node to its polygon center	$\sqrt{\frac{2}{3}}$
distance from vertex to sphere center	1
volume of block	$\frac{8}{3\sqrt{3}} = 1.53960$

The third block has twelve faces, each face is a equal-lateral pentagon. The block is in the unit sphere. The distance from each vertex to the sphere center is 1.

Third block of 12 face polygons

boundary polygon number	12
edge number of polygons	5
angle of two adjacent polygon planes	$180 - arctan(2)$
edge length of polygons	$\frac{1}{\sqrt{3}}(\sqrt{5} - 1)$
distance from polygon to sphere center	$\frac{1}{\sqrt{15}}\sqrt{5 + 2\sqrt{5}}$
distance of node to its polygon center	$\frac{1}{\sqrt{15}}\sqrt{2(5 - \sqrt{5})}$
distance from vertex to sphere center	1
volume of block	$\frac{2}{3}\frac{1}{\sqrt{3}}\sqrt{10(3 + \sqrt{5})} = 2.78516$

In order to use the simplex integration, the node coordinates (x, y) in each boundary

polygon have to be computed. Based on the angles of boundary planes and distances of these planes to the sphere center, the polygons, the edges and the vertices can be computed.

The three dimensional simplex integrations give the same volume as computed directly from the geometric formulae listed in the tables.

Rock Failure Examples by DDA

The equilibrium of the DDA method is reached by minimizing the total potential energy. As the energy is computed by integrations, most of the DDA formulae are formed by the polynomial integrations over the generally shaped blocks. The simplex integration is developed and applied to DDA formulation. This new integration gives analytical solutions. The integrand can be multi-dimensional polynomials.

In the two dimensional case, the integration domain can be any convex or concave polygons. In the three dimensional case, the integration domain can be any complex body with plane polygon boundary. The integration results are simply represented by the coordinates of boundary vertices. Based on the simplex integration, DDA algorithms are simple, efficient yet accurate. Most important, the accurate integral solution of mass matrix ensured the convergence of "open-close" iterations.

Three rock failure examples are presented. The failure process is a transition from continuous to discontinuous states. The discontinuous deformation analysis (DDA) has to fulfill physical laws of both continuous and discontinuous materials. When the computed displacements and deformations are large enough to be visible, the mechanism of the failure and the final damage can be shown, and the ultimate strength of materials or structures can be intuitively estimated. The visible sliding and joint opening from the computation can demonstrate that the physical laws are satisfied.

The computations require equilibrium in both the discontinuous contacts and the continuous zones throughout the entire dynamic process. Following a real time sequence, the DDA uses a step by step approach. The displacements of each time step are so small that normal linear equations for small displacements can be adopted. At the end of each time step, the equilibrium in both discontinuous interfaces and continuous zones are reached.

As the step displacements are small, the kinematic relation and friction law are expressed as linear inequalities. Based on natural contact phenomena, an "entrance theory"

was developed. The entrance lines are used to form linear inequality equations. The same linear inequalities are used to define the entrance distances, entrance points, entrance constraints and entrance criteria. The "open-close" iterations ensure that no tension and no penetration occur at all entrance positions. There are three entrance modes: open, sliding and locking. Coloumb's Law is also fulfilled at all entrance modes and all entrance positions.

There are 1500 to 2000 rock blocks in each example. The dimensions of computed regions are about 40 to 80 meters. The numbers of time steps are from 300 to 600. The total elapsed times are from 0.2 to 2.0 second. The maximum total displacements are more than ten times the average block diameter.

Figure 5 shows the collapse process of a tunnel caused by high initial stresses.

Figure 6 shows the penetration of a missile at a velocity of 300 meters per second into a blocky rock mass with two tunnels.

Figure 7 shows the damage state as a strong stress wave passing through three tunnels excavated in a blocky rock mass.

Figure 8 shows a rock toppling failure caused by slope excavation.

Acknowledgement

References

Shi, G. (1993). "Block system modeling by Discontinuous Deformation Analysis." Southampton UK and Boston USA.

Shi, G. (1994). "Modeling dynamic rock failure by Discontinuous Deformation Analysis with simplex integrations." *Proceedings of the 1st North American Rock Mechanics Symposium (Austin, Texas)*, 591-598.
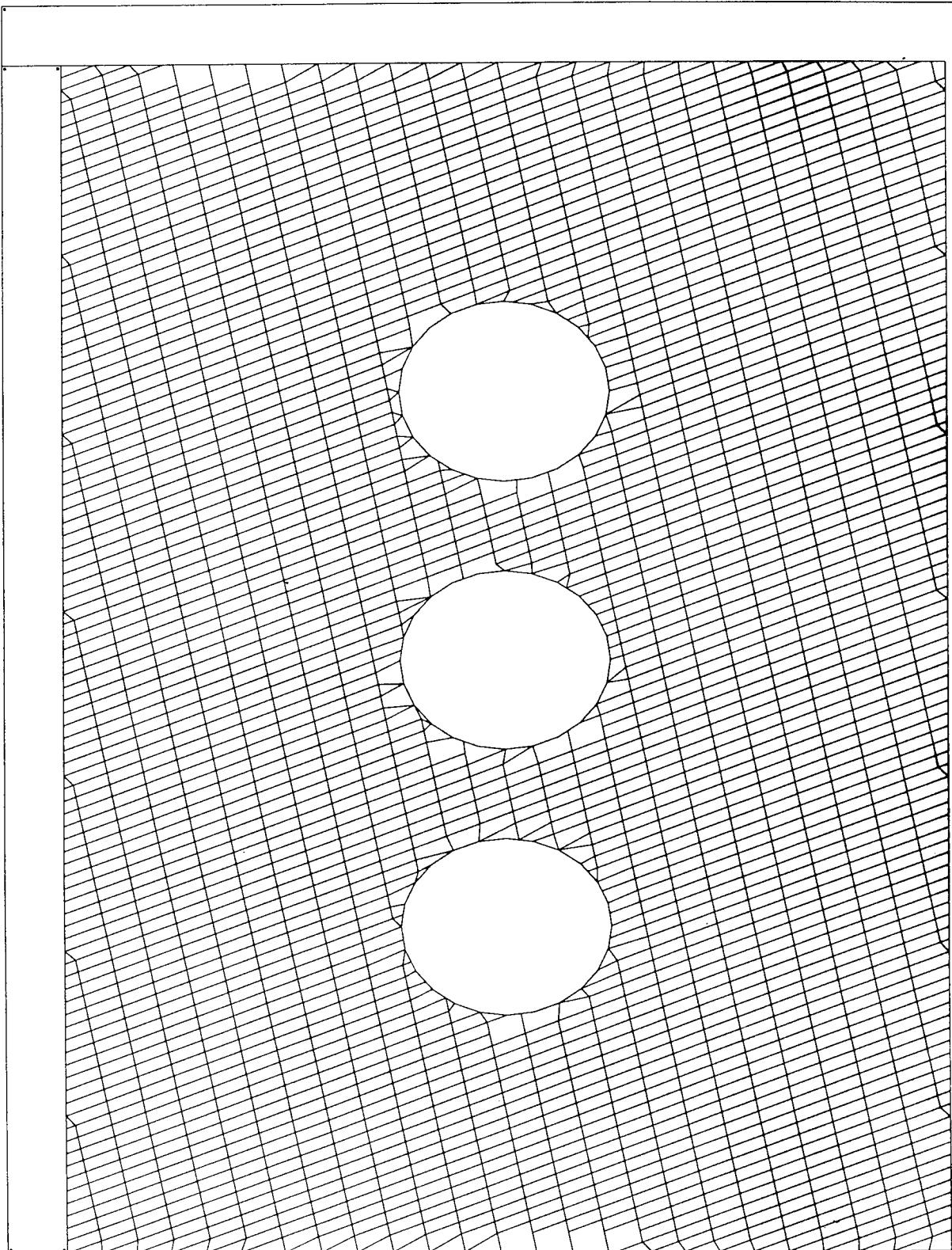
Figure 5. Collapse of a tunnel caused by high initial stresses (continued)
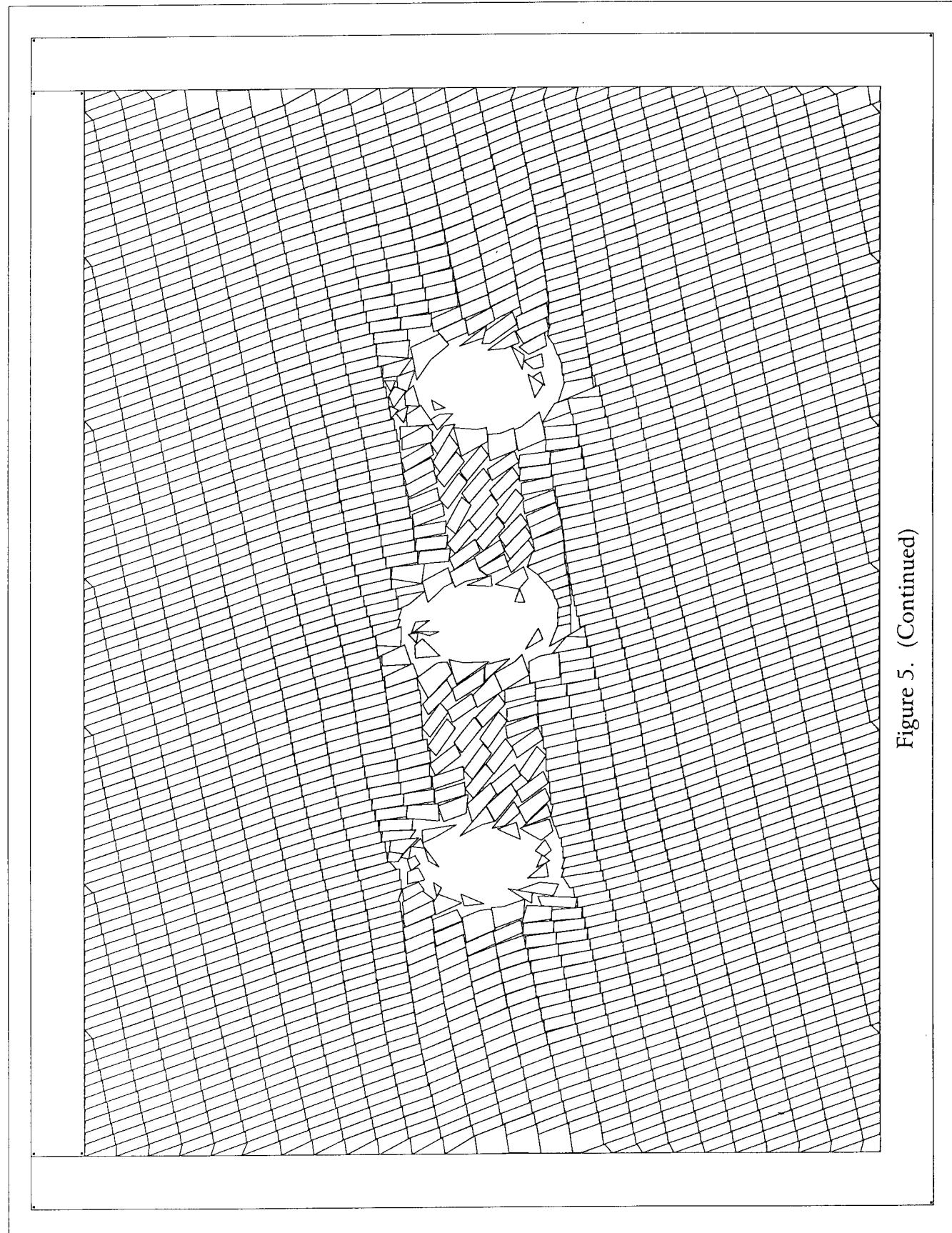
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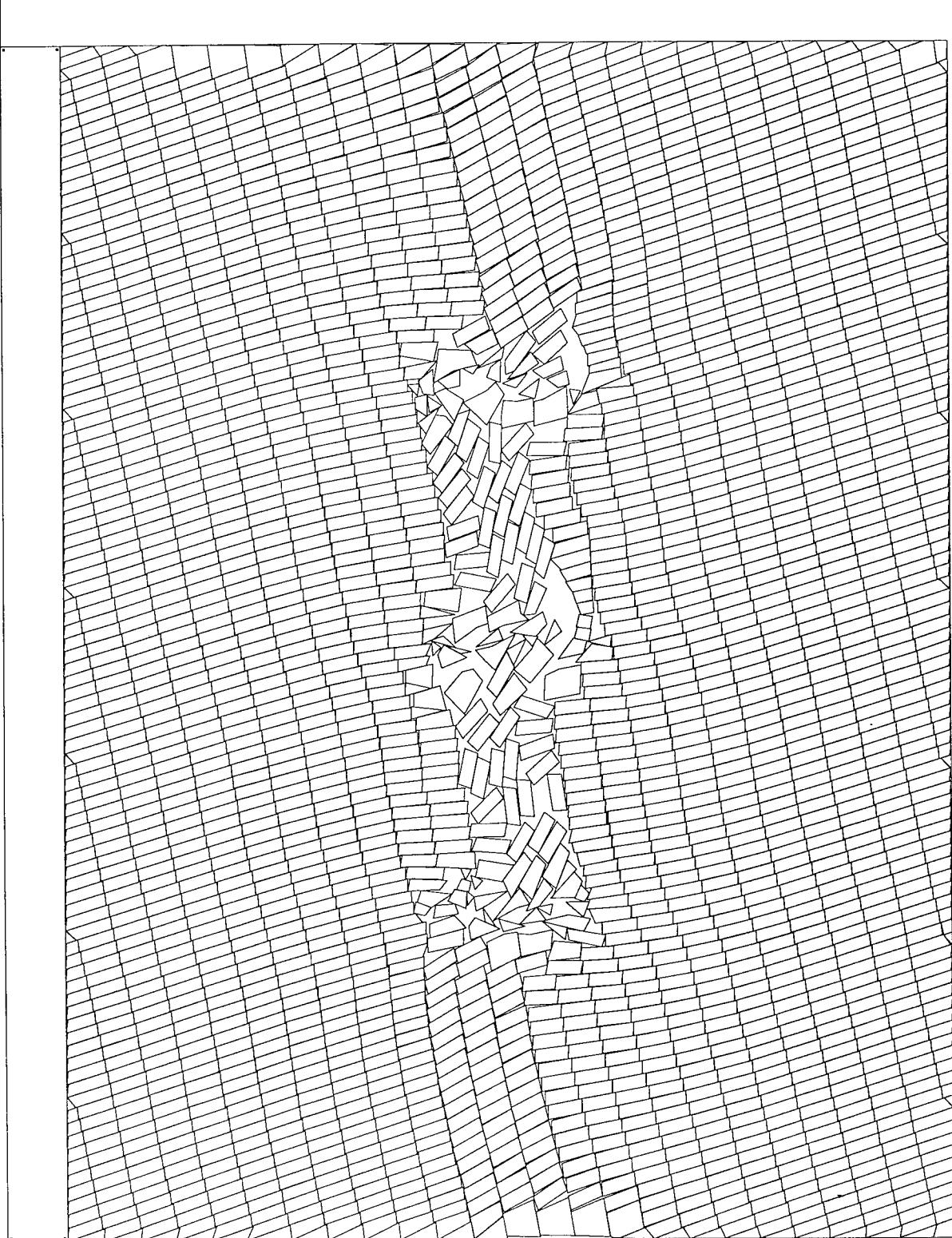
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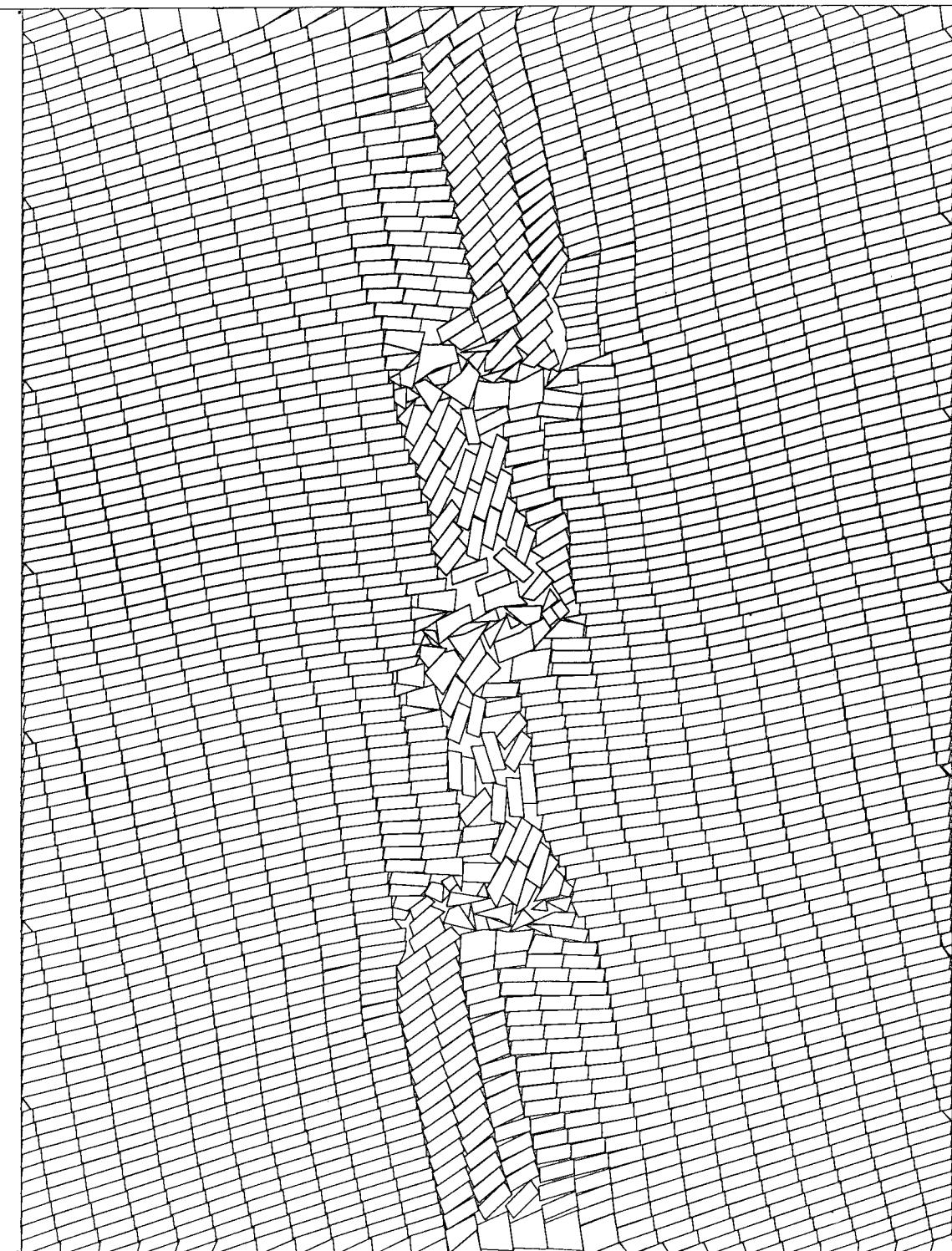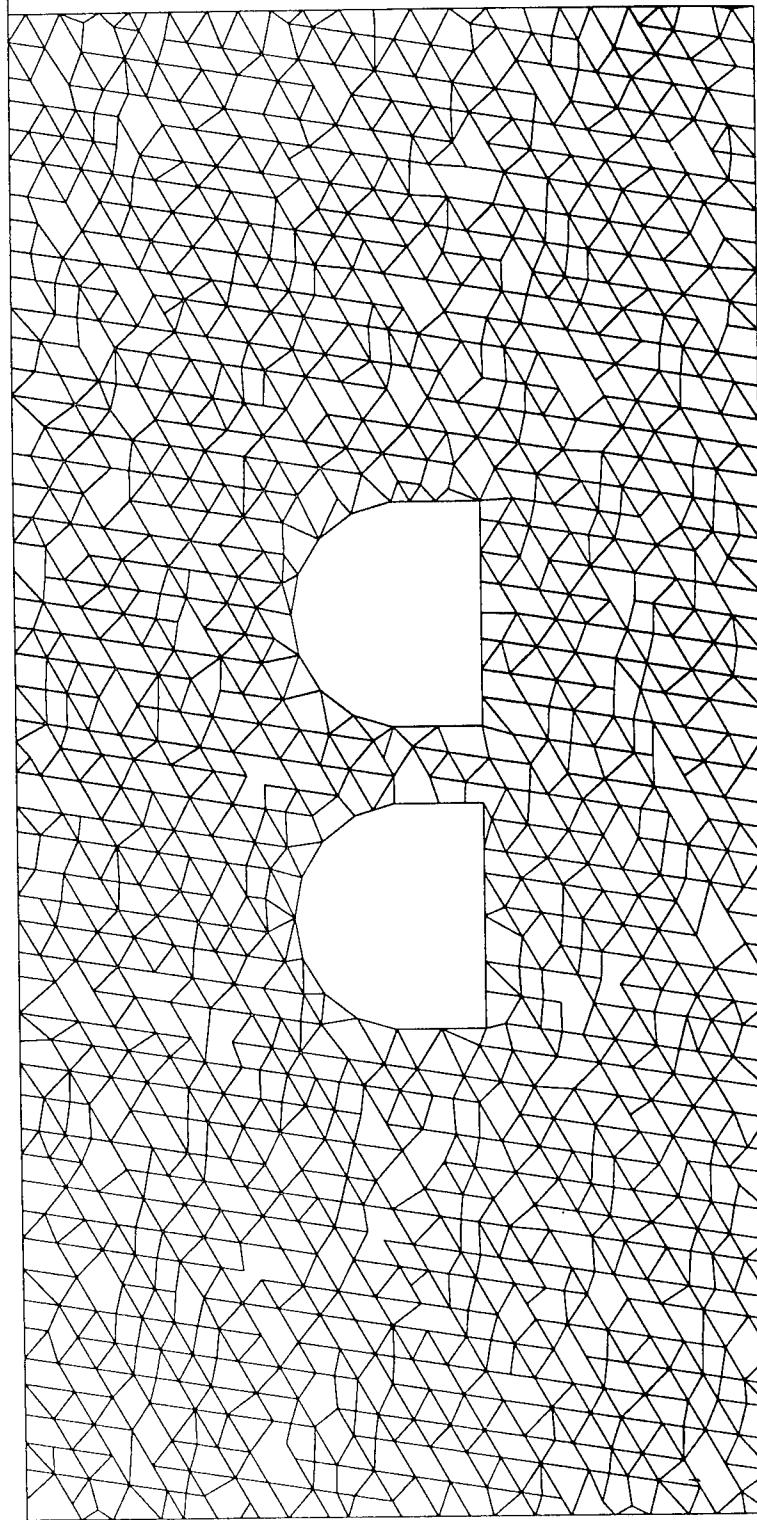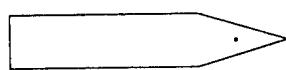
Figure 6. The penetration of a missile at a velocity of 300 meters per second into a blocky rock mass with two tunnels (continued)
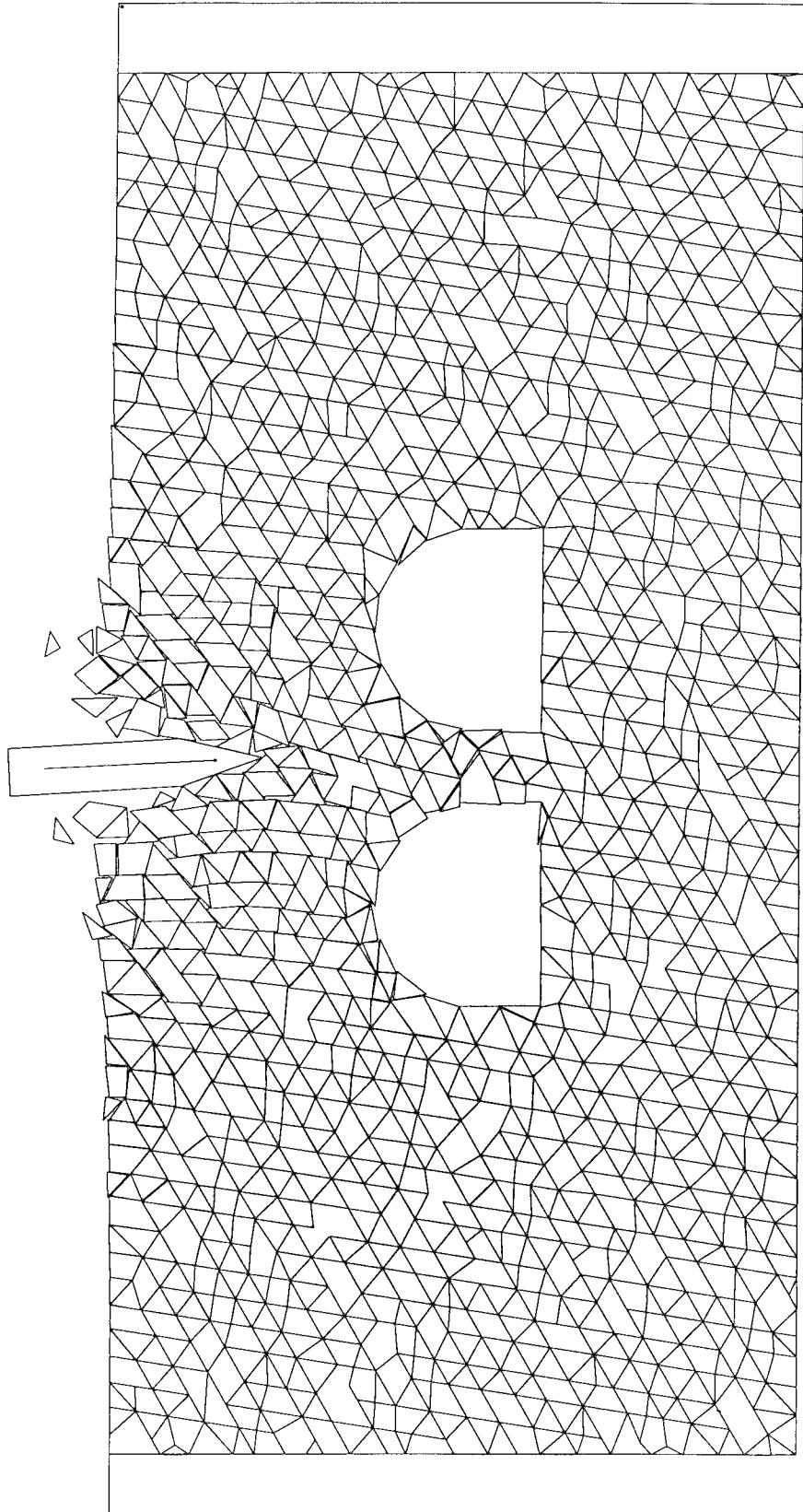
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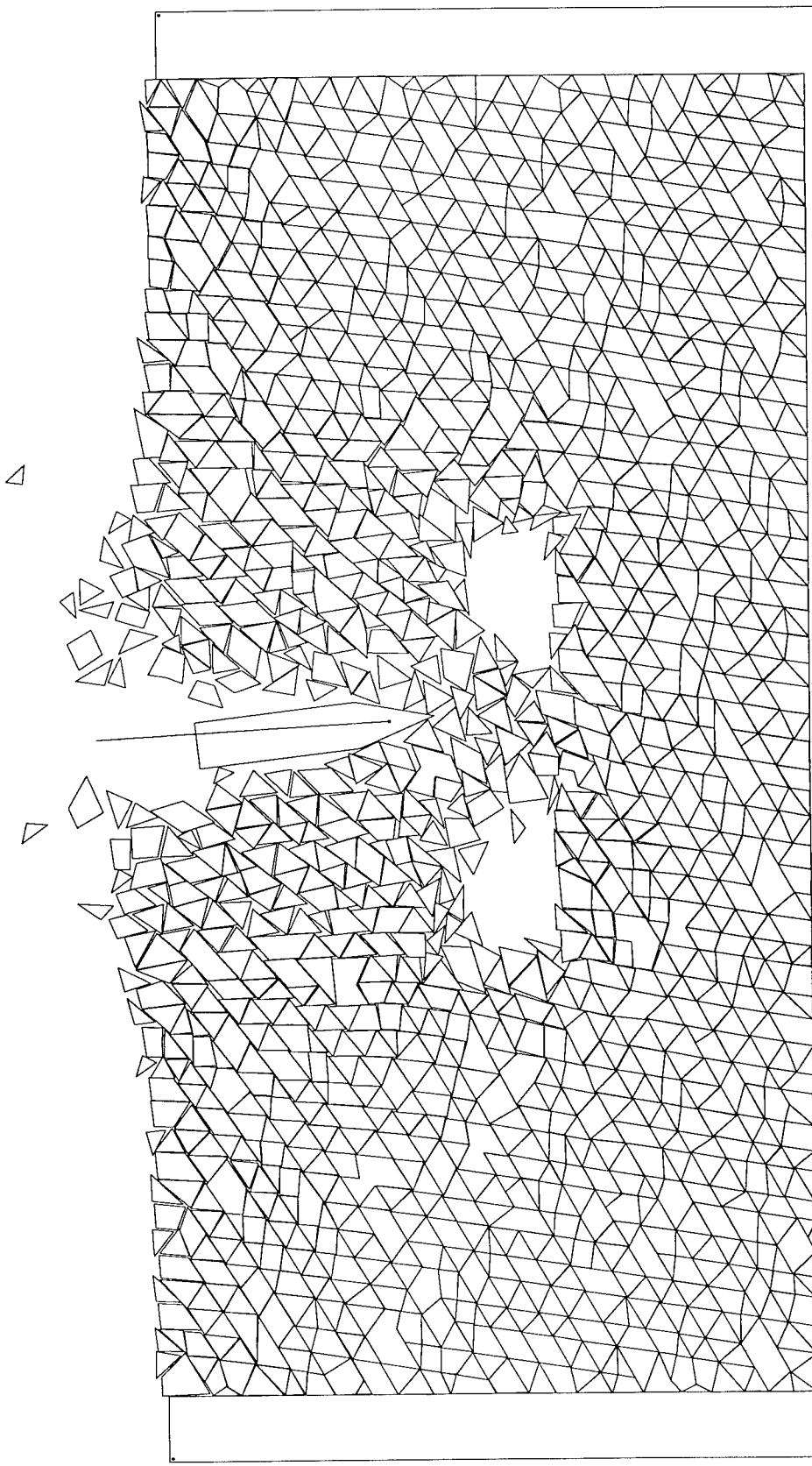
Figure 6. (Continued)

Figure 6. (Concluded)
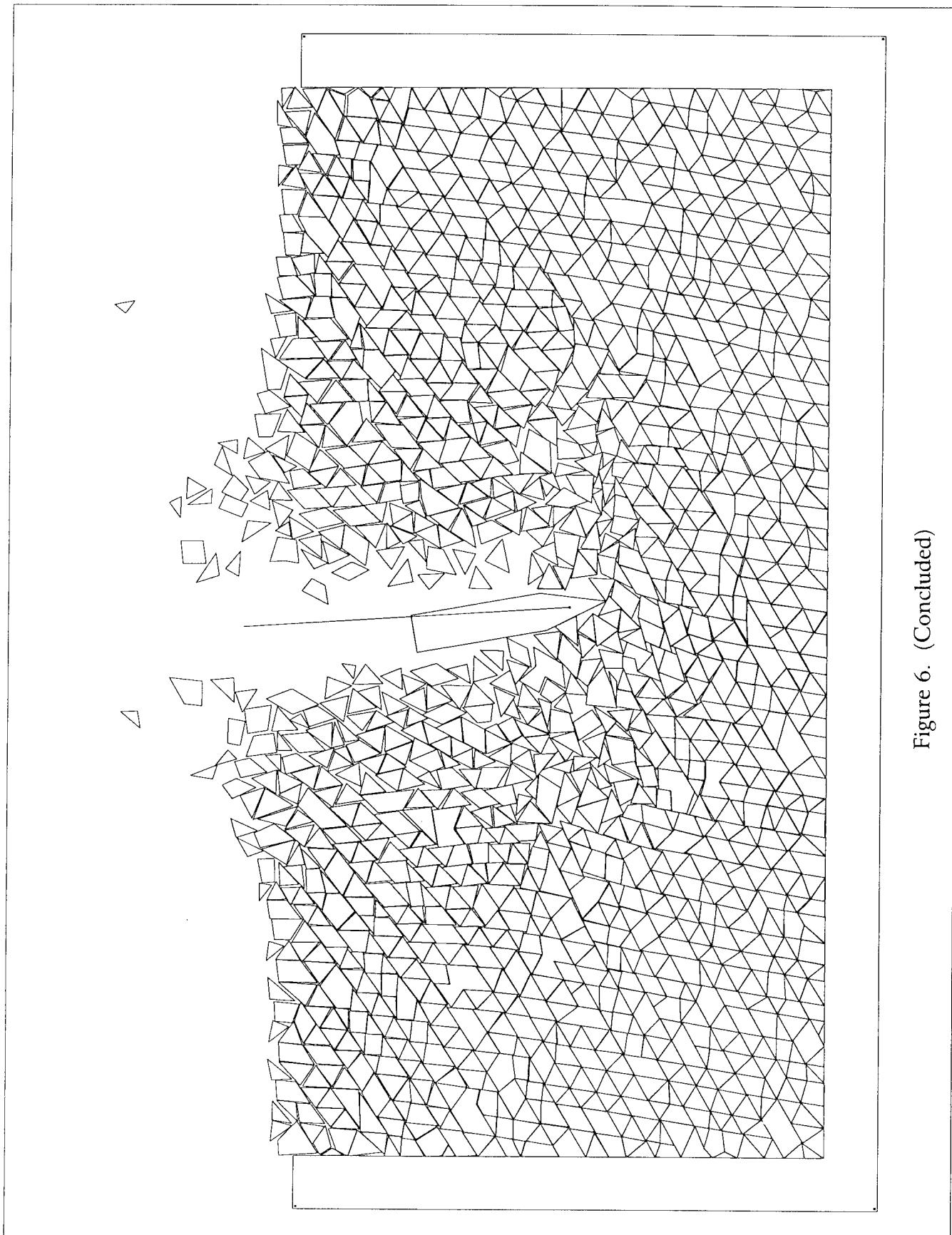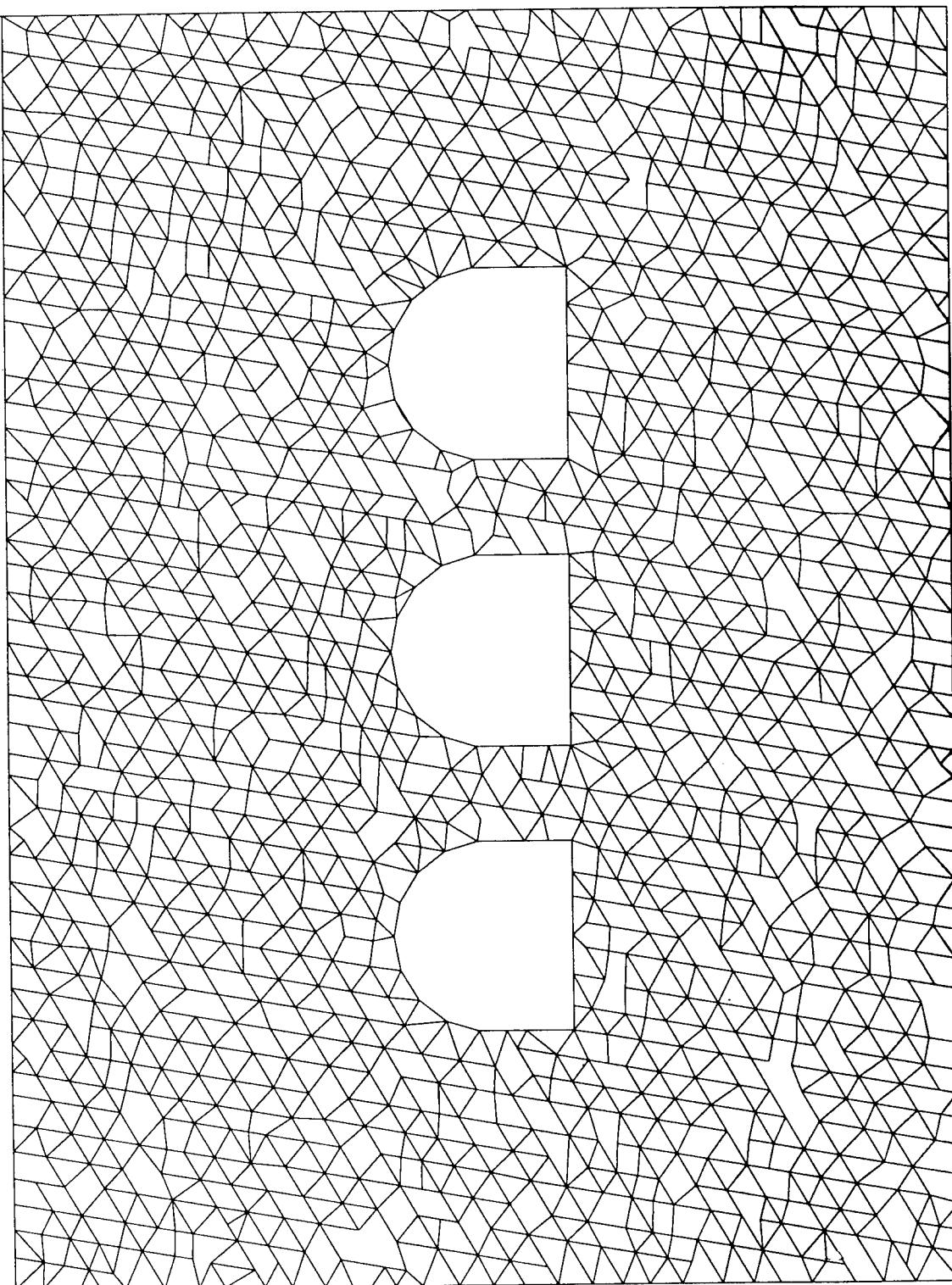
Figure 7. A strong stress wave passing through three tunnels excavated in a blocky rock mass (continued)

Figure 7. (Continued)
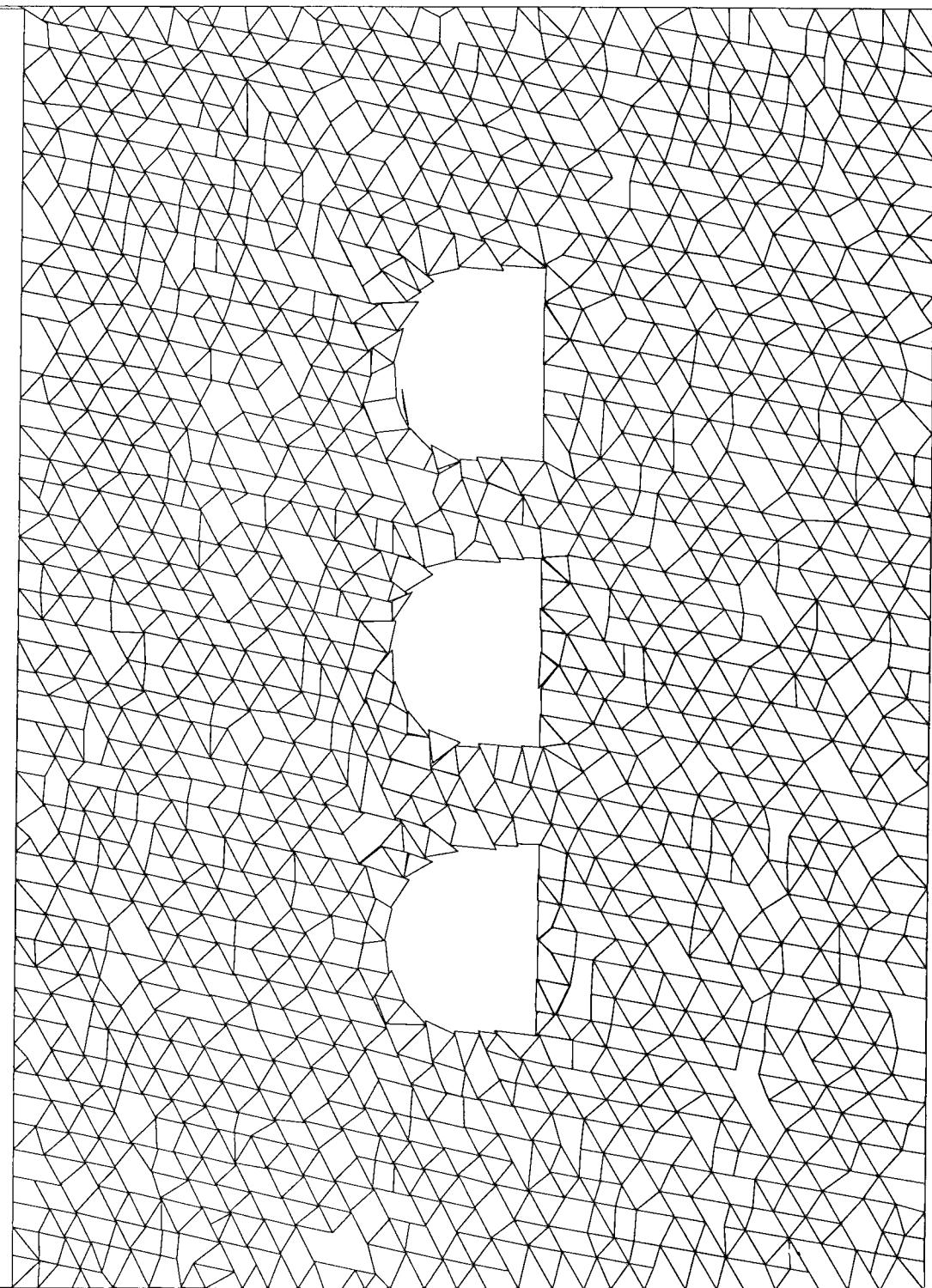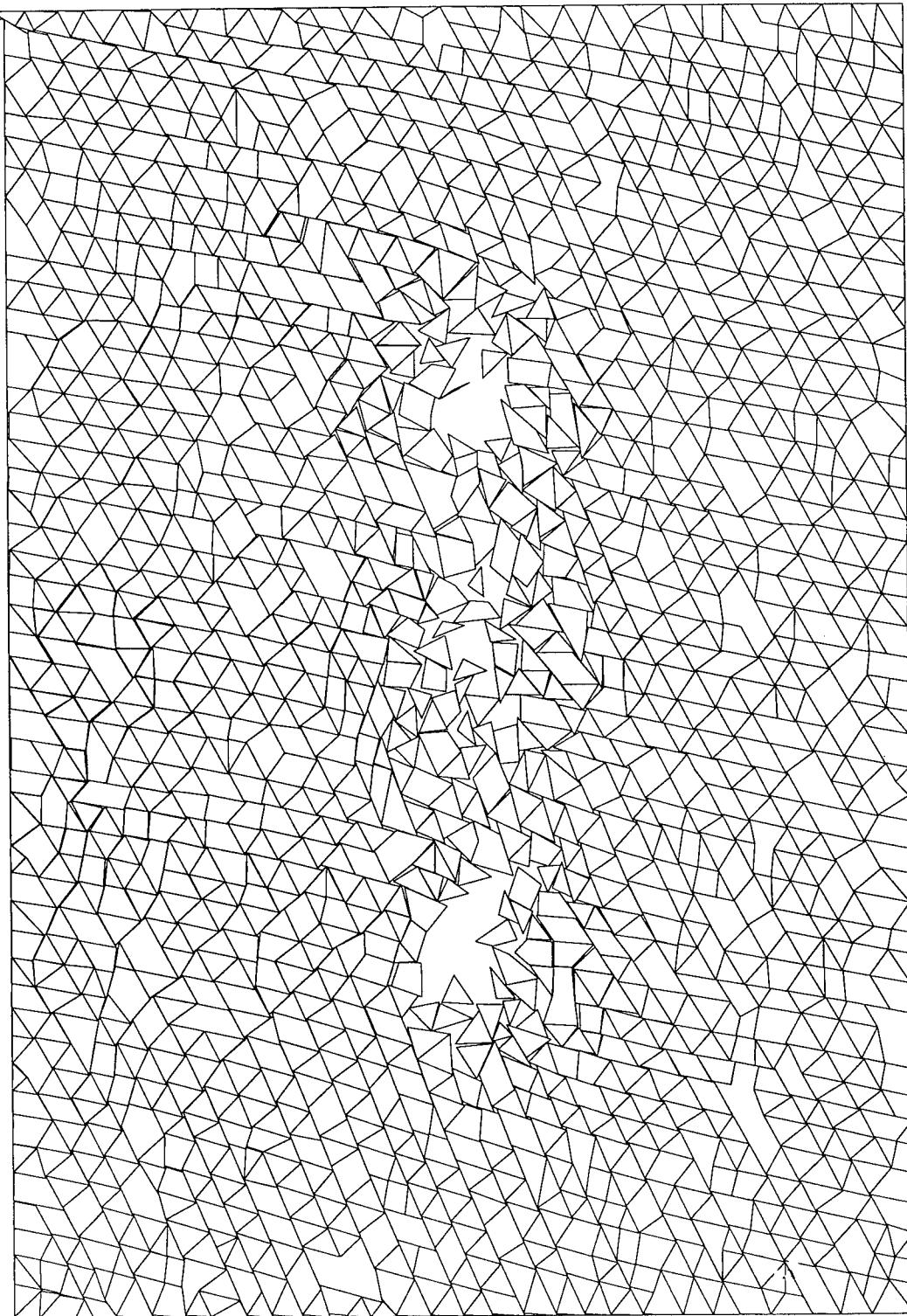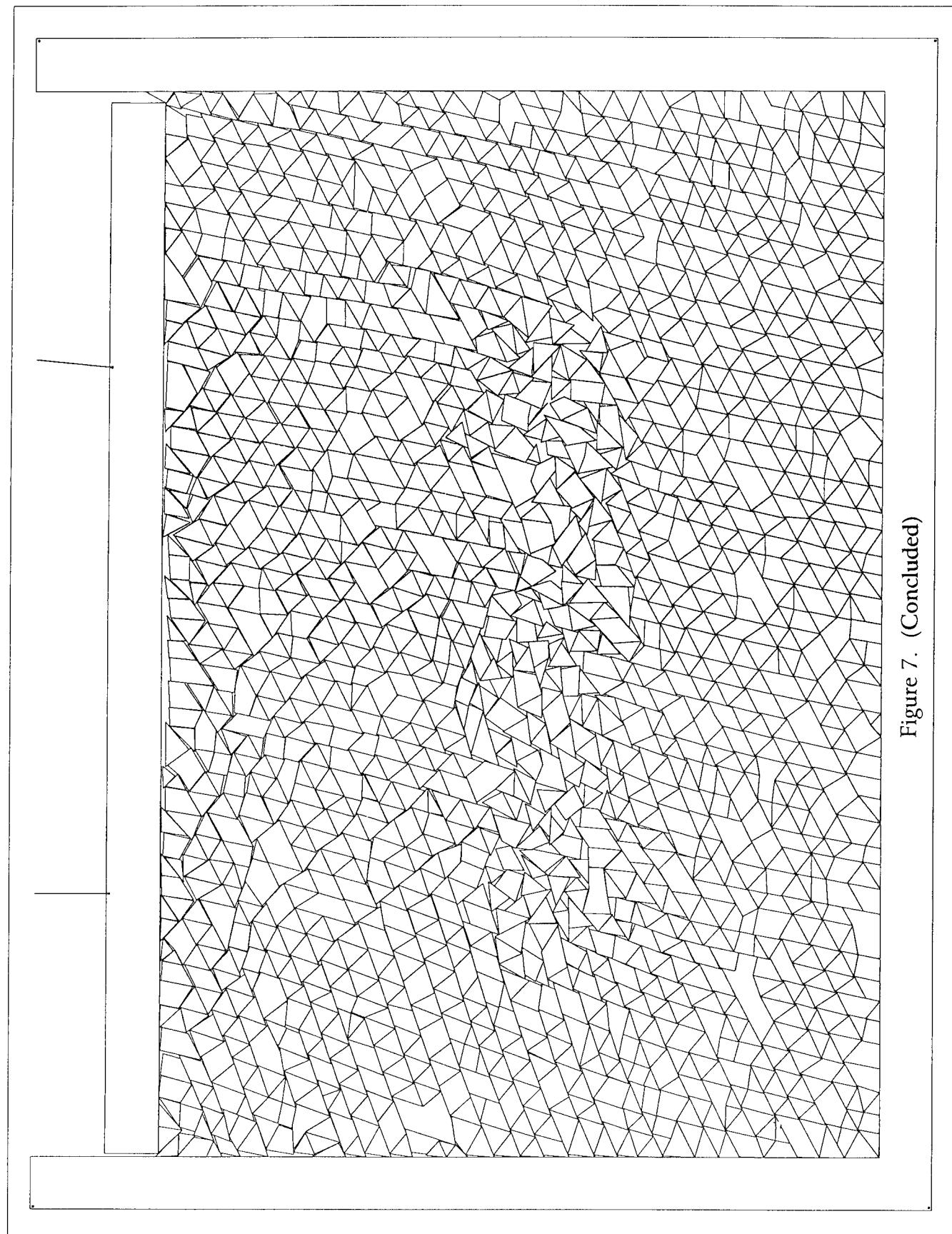
Figure 7. (Continued)

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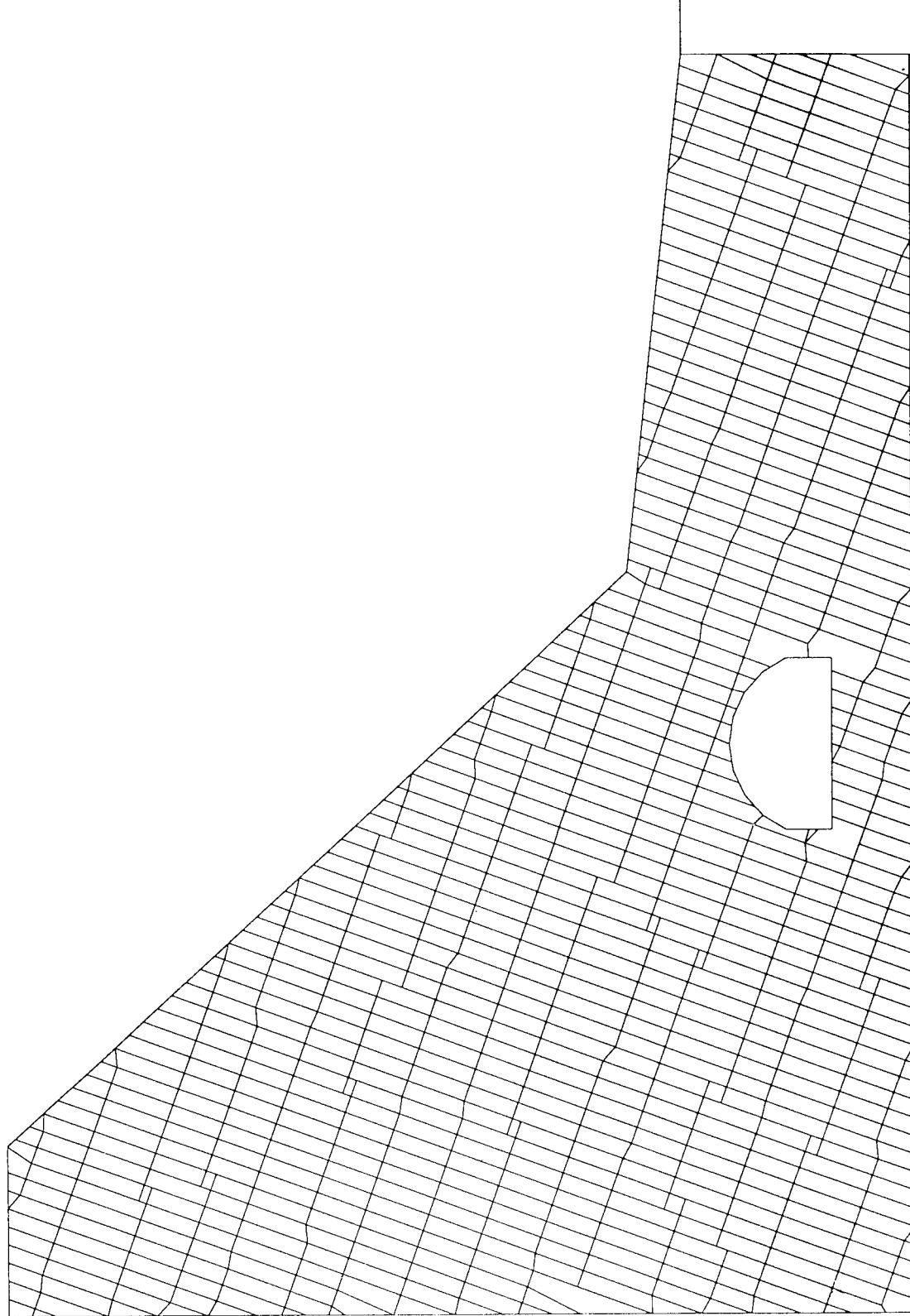
Figure 8. A rock toppling failure caused by slope excavation (continued)

Figure 8. (Continued)
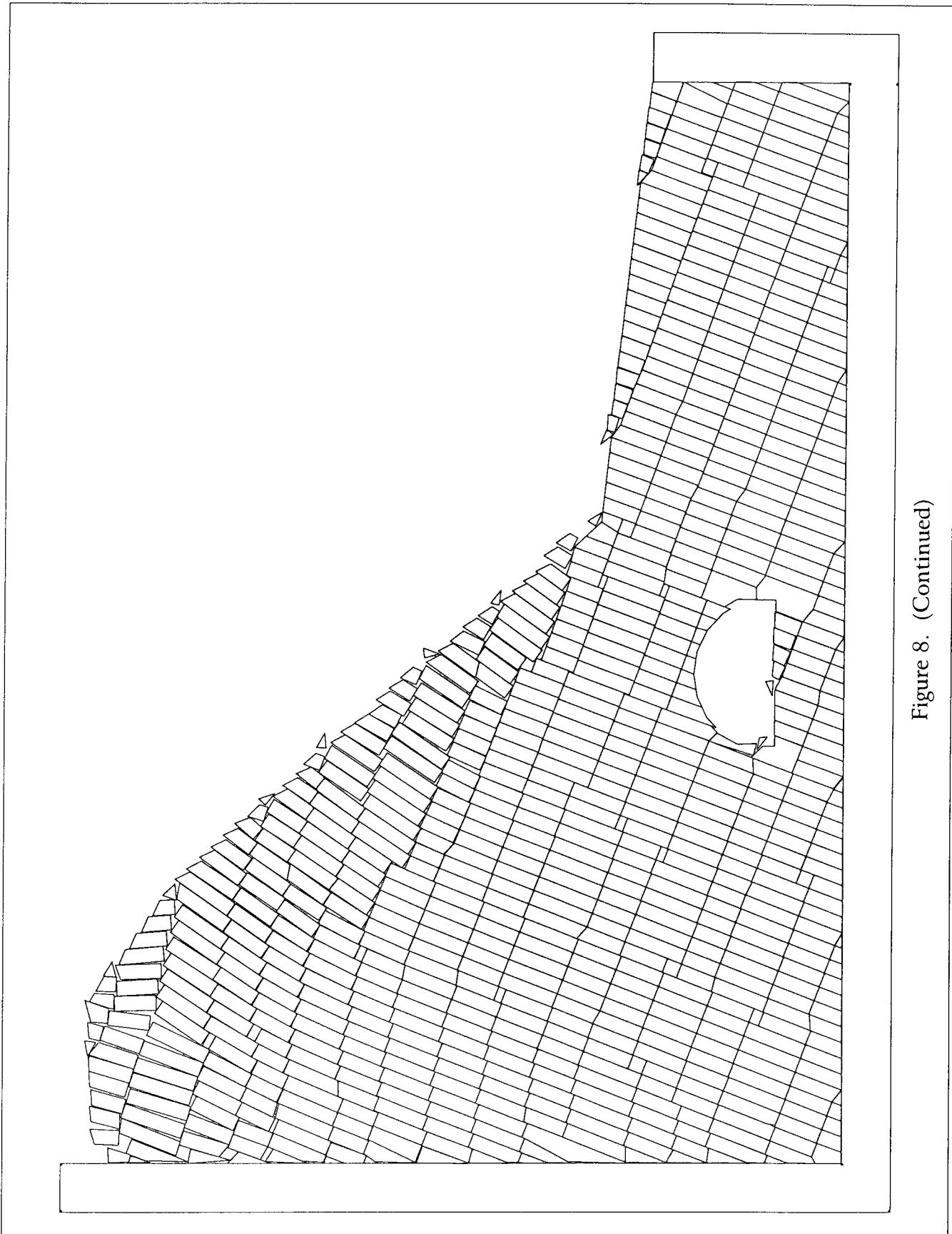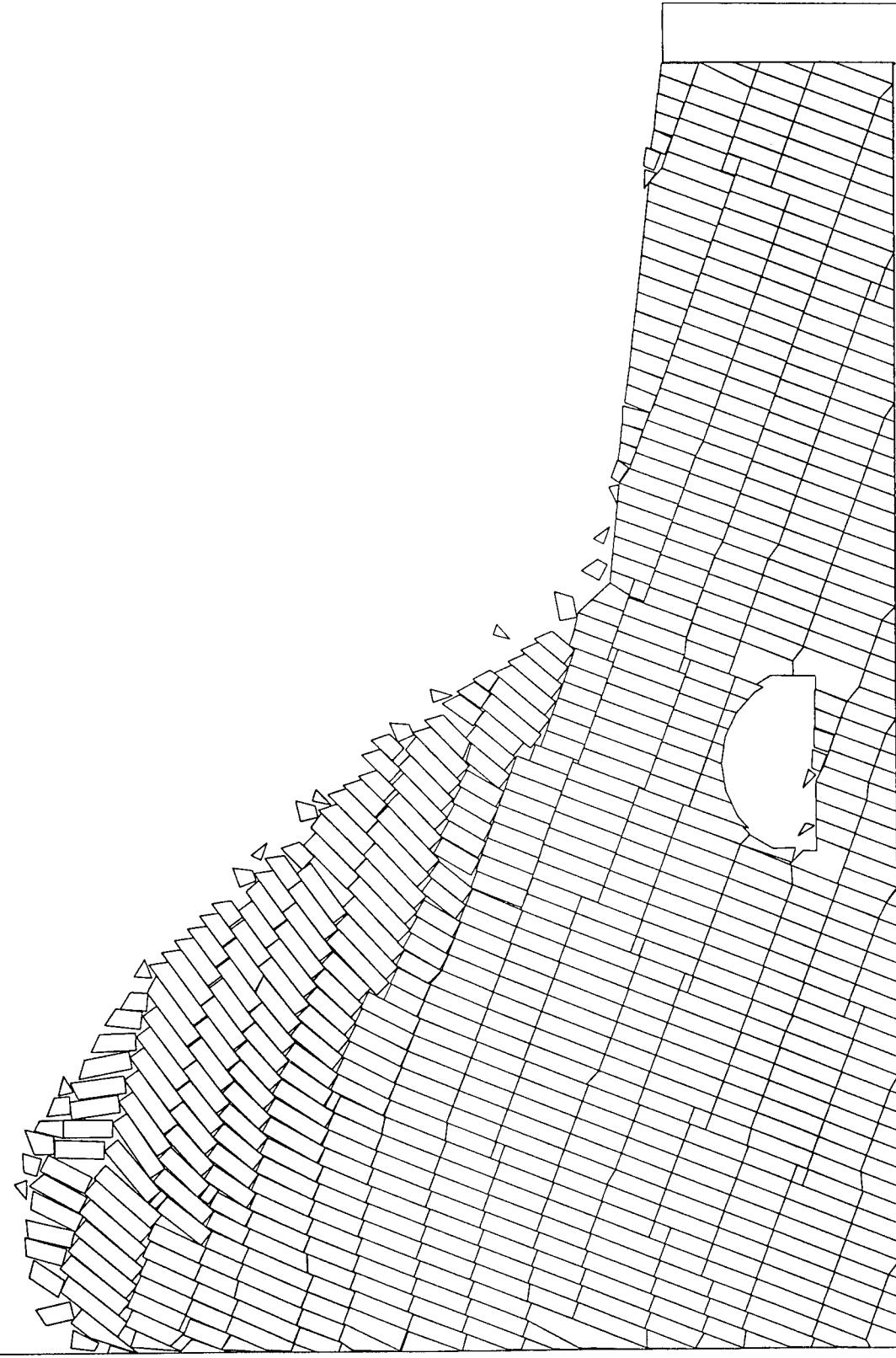
Figure 8. (Continued)

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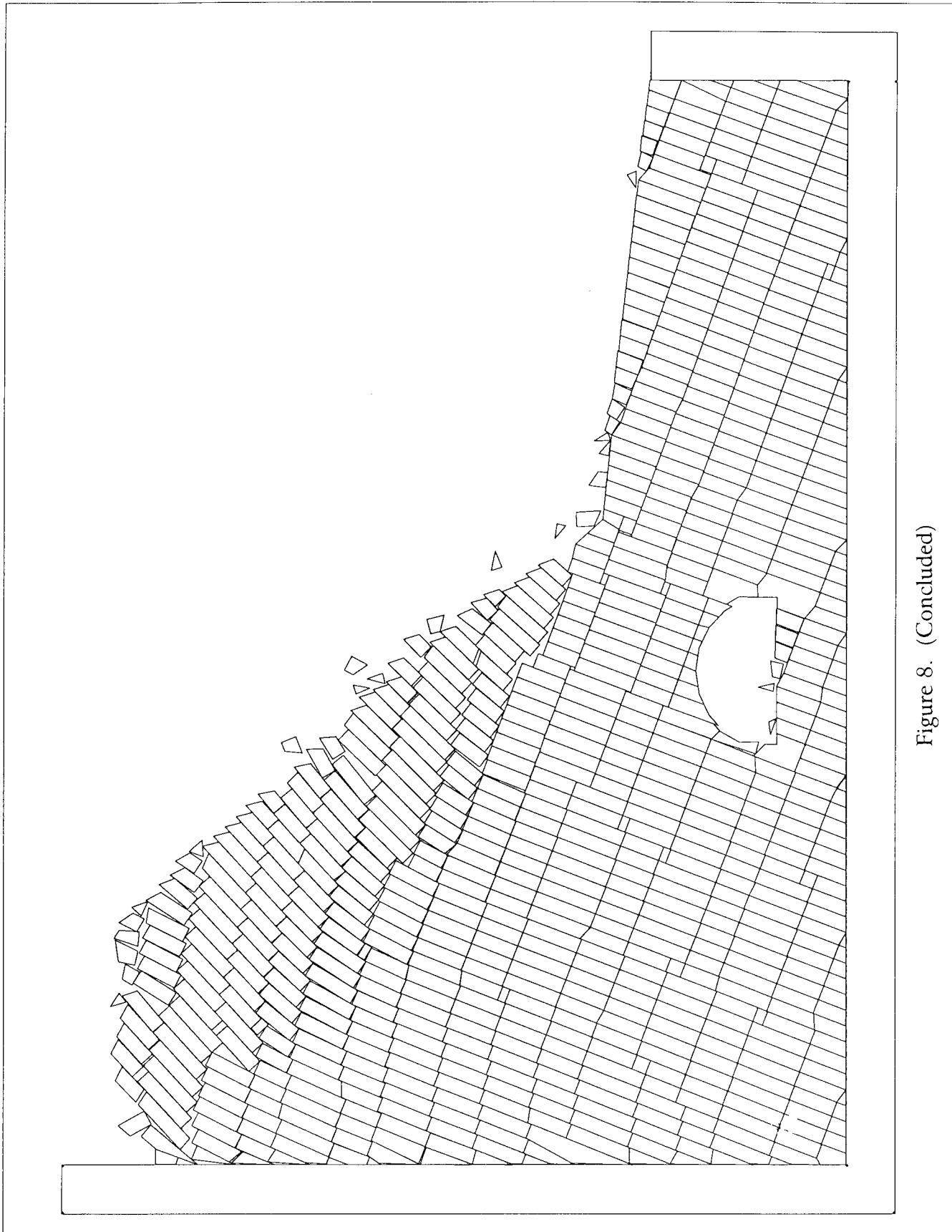
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13.ABSTRACT *(Maximum 200 words)*

Aiming at global analysis, the well known mathematical manifold is perhaps the most important subject of modern mathematics. Based upon mathematical manifold, this numerical manifold method is a newly developed general numerical method. This method computes the movements and deformations of structures or materials. The meshes of the numerical manifold method are finite covers. As the material domains, the finite covers overlapped each other and covered the entire material volume. On each cover, the manifold method defines an independent cover displacement function. The cover displacement functions on individual covers are connected to form a global displacement function on the entire material volume.

The global displacement function are the weighted averages of local independent cover functions on the common part of several covers. Using the entire finite cover systems, continuous, jointed, or blocky materials can be computed in a mathematically consistent manner. For a manifold computation, the mathematical mesh and physical mesh are independent. Therefore, the mathematical mesh is free to define and free to change. As with the mathematical mesh, the covers can be moved, split, removed, and added. By moving the covers, the large deformations and moving boundaries can be computed

(Continued)

14.SUBJECT TERMS

Continuum	Large deformation	Moving boundaries
Discontinuum	Mathematical manifold	Stress analysis
Global analysis	Mathematical mesh	Physical mesh

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by steps. By dividing a cover to two or more independent covers with their displacement functions, jointed and blocky materials can be modeled.

Both the finite element method (FEM) for continua and the discontinuous deformation analysis (DDA) for block systems are special cases of this numerical manifold method. In the current development stage of numerical manifold method, by using finite cover approach, the extended FEM can compute more flexible and visible deformations and movements of joints and blocks.